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Taylor operations on finite reflexive structures

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Abstract

In [25], a discrete homotopy theory for reflexive digraphs was developed. In the present paper, we prove that if a finite, connected reflexive digraph X has non-trivial homotopy in some dimension but none of its proper retracts does, then the digraph is idempotent trivial. This is used to prove that if a structure admits an operation satisfying specific identities (a *Taylor operation*), then every homotopy group of X is trivial. We give several applications of this result to the study of the algorithmic complexity of constraint satisfaction problems: we prove dichotomy results for the retraction problem for various families of digraphs.

1 Introduction

The last 5 years have seen a flourish of activity surrounding the dichotomy question for the algorithmic complexity of constraint satisfaction problems (CSP's), motivated in part by a successful connection linking the complexity of CSP's and the structure of their associated (universal) algebra, first uncovered by P. Jeavons [19] and later developed by A. Bulatov, P. Jeavons and A. Krokhin [5]; the present paper is a modest contribution in this direction.

Let Γ be a set of finitary relations on a nonempty finite set A. The *(restricted) constraint satisfaction problem* $CSP(\Gamma)$ is the following decision problem: given a pair $\langle V, C\rangle$ where V is a finite set of variables and C is a finite set of constraints, i.e. pairs (s, θ) where s is a tuple of variables from V and $\theta \in \Gamma$, determine if there exists a function $f : V \to A$ such that, for each constraint (s, θ) we have that $f(s) \in \theta$. A wide range of standard decision problems are of this form, such as graph q-colouring, 3-satisfiability, Horn-SAT, graph unreachability and so on. It is immediate that for any Γ the problem $CSP(\Gamma)$ is in the class **NP**, and that the general problem, where Γ consists of all possible relations on A, is **NP**-complete. The question that poses itself immediately is to determine for which sets Γ the problem is tractable; in other words, what kind of restrictions on constraints will ensure that a constraint satisfaction problem admits a polynomial-time algorithm?

In 1979, Schaefer obtained a dichotomy result for CSP's on a 2-element universe [34]: if Γ is any set of relations on $\{0, 1\}$, then $CSP(\Gamma)$ is either in **P**

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or **NP**-complete; he also characterised completely the nature of the problems in the two classes of problems. By a well-known result of Ladner, if $\mathbf{P} \neq \mathbf{NP}$ then there are problems which are neither in **P** nor **NP**-complete. However, Schaefer's result shows that these problems cannot be Boolean CSP's. In fact, T. Feder and M. Vardi [15, 16] have conjectured that this must hold for all CSP's.

In 1998, P. Jeavons [19] made a striking observation, showing that Schaefer's classification of tractable Boolean CSP's precisely matched that of so-called *mini clones*, a well-known universal algebraic result of I. G. Rosenberg [33] whose Boolean counterpart could already be found in Post [31]. Based on this result of Jeavons, Bulatov, Jeavons and Krokhin cleared up the connection between CSP's and universal algebras and formulated a precise dichotomy conjecture. We present here a modified but equivalent version. We refer the reader to [18], [29] and [36] for basic results in universal algebra, [30] for computational complexity and [35] for algebraic topology.

We say that the n-ary operation $f : A^n \rightarrow A$ on A *preserves* the k-ary relation θ on A, or that θ is *invariant* under f if the following holds: given any matrix M of size $k \times n$ whose columns are in θ, the column obtained by applying f to the k rows of M is also in θ. Given a set Γ of relations on A, let F denote the set of all operations on A that preserve every member of Γ. The (universal) unindexed algebra $\mathbb{A}(\Gamma) = \langle A; F \rangle$ with universe A and set of basic operations F essentially determines the complexity of the problem $CSP(\Gamma)$: if two sets of relations Γ and Γ' are such that their associated algebras are "equivalent", then the problems $CSP(\Gamma)$ and $CSP(\Gamma')$ are poly-time equivalent.

In [5] it is shown that, via poly-time reductions, we may safely restrict our attention to *retraction problems*, i.e. CSP's with sets Γ that contain all singleton unary relations $\{a\}$, $a \in A$; these are the sets of relations such that all basic operations f of their associated algebra are *idempotent*, i.e. satisfy the identity $f(x, x \dots, x) \approx x$ (where the $\approx$ symbol indicates that all variables are universally quantified.) An algebra is idempotent if all its basic operations are idempotent. An n-ary idempotent operation f on A is *Taylor* if it satisfies the following condition: for every $1 \leq i \leq n$, f satisfies an identity of the form

$$f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n) \approx f(y_1, \dots, y_{i-1}, y, y_{i+1}, \dots, y_n)$$

where $x_j, y_j \in \{x, y\}$ for all $1 \leq j \leq n$ (see [37], [18]). Taylor operations first appeared in [37] in connection with topological algebras; they have since come to play an important rôle in the classification of varieties of universal algebras (see [18]). An operation obtained by composing basic operations of an algebra is called a *term* operation of the algebra.

The following hardness criterion was first proved in [5] in a different formulation, but may be found in the present form in [28].

Theorem 1.1. *Suppose the algebra $\mathbb{A}(\Gamma)$ is idempotent. If $\mathbb{A}(\Gamma)$ admits no Taylor term then the problem $CSP(\Gamma)$ is* **NP***-complete.*

It is conjectured that, in essence, this is the only reason why a constraint satisfaction problem is hard:

Dichotomy Conjecture 1 ([5])**.** *Suppose the algebra* $\mathbb{A}(\Gamma)$ *is idempotent. If* $\mathbb{A}(\Gamma)$ *admits a Taylor term then* $CSP(\Gamma)$ *is in* $\mathbf{P}$*; otherwise it is* $\mathbf{NP}$*-complete.*

This approach has proved quite fruitful, and various special cases of the conjecture have been obtained, most notably the 3-element case [3], and the case of algebras admitting a Mal'tsev term [2], see also [7]. The conjecture has also been proved in the case of so-called *list homomorphism problems*, i.e. when the set Γ contains all unary relations [4].

In the case where Γ consists of a single binary relation together with all unary relations, we get so-called *graph list homomorphism problems*. T. Feder, P. Hell and J. Huang have obtained nice characterisations of the tractable and **NP**-hard cases, for various types of graphs (see [14] and also [13]). In essence, the general approach consists of a sieve, where one eliminates hard cases, say by characterising those structures that do not admit a compatible Taylor operation, and proving that the remaining structures have nice enough properties to ensure that a polynomial-time algorithm exists. One of the difficulties in the above approach resides in determining which graphs admit a Taylor operation (the other being, of course, of exhibiting a poly-time algorithm for the appropriate structures.)

Given a set of relations Γ on A, we say that the relation θ is *inferred* from Γ if every term operation of the algebra $\mathbb{A}(\Gamma)$ preserves θ. Equivalently, the k-ary relation θ is inferred from Γ if it can be expressed in the form

$$\mu = \{(a_1, \dots, a_k) : \phi(a_1, \dots, a_k) \text{ holds}\}$$

where $\phi(x_1, \dots, x_k)$ is a first-order, primitive positive formula with free variables $x_1, \dots, x_k$, and the relations appearing in ϕ are all in Γ. Obviously, if one can infer from Γ a relation θ which is invariant under no Taylor operation, then the problem $CSP(\Gamma)$ is **NP**-complete. Not surprisingly, in view of the conjecture, all known hardness results for CSP's are essentially obtained by producing the appropriate inferred relation.

In [16], Feder and Vardi conjecture that all tractable CSP's should fall into two distinct families of problems, the *group-like* and the *bounded width* problems. In [26] it is shown that the associated algebra of a CSP of bounded width must admit very special idempotent terms, and in particular a Taylor term. It follows that a CSP admitting no Taylor operation cannot be of bounded width: not only is the problem presumably hard (**NP**-complete), there is a large class of poly-time algorithms that provably cannot solve such a problem (see [26] for applications.) So although conjecturally the two concepts coincide, it is of independent interest to prove not only that a problem is **NP**-complete but furthermore that it admits no Taylor operation.

Let θ be a reflexive binary relation on the set A, i.e. $(x, x) \in \theta$ for all $x \in A$. We say that the pair (A, θ) is a *binary reflexive structure*, or *reflexive digraph*, or more succinctly just a *structure* or a *digraph*. In [25], a discrete homotopy theory was developed for these structures; to each pointed structure is associated a family of (homotopy) groups, which behave somewhat as if the structure were a topological space and the structure-preserving maps were continuous:

we review the necessary definitions, notations and results in section 2.1. The main result of this paper is Theorem 2.3: *if a finite, connected, binary reflexive structure admits a Taylor operation then all its (discrete) homotopy groups are trivial*. In collaboration with L. Zádori [28] we had already proved a similar result in the special case of posets, i.e. when the binary relation is also antisymmetric and transitive. Even though part of the argument is similar, the present proof is quite a bit more involved, since reflexive structures do not have the natural topological structure that posets have. In fact, we will deduce our main result from the following more general statement: if in some dimension a finite, connected, binary reflexive structure has non-trivial homotopy but none of its proper retracts does, then this structure admits no non-trivial idempotent operation of any arity (Theorem 2.11). This theorem is a wide-ranging generalisation of various known results about so-called *projective* or *idempotent trivial* graphs and posets, see [6], [12], [10], [11], [17], [32]. In particular, we prove in section 3 that any reflexive graph or poset that triangulates a sphere must be idempotent trivial (Theorem 3.3).

Because reflexive digraphs are easier to construct than posets as inferred relations, the present result has a wider range of applications (see for example [24]). In section 3, we shall use this criterion to prove various hardness and dichotomy results for CSP's. In particular, we generalise to intransitive digraphs an unpublished result of Feder and Hell [13] on directed cycles (Theorem 3.6). We also prove a dichotomy result for tournaments (Theorem 3.7) and for series-parallel structures (Theorem 3.8).

2 Main result

2.1 Preliminaries and statement of the main result

We now review all the basic definitions, notations and results concerning binary reflexive structures and the discrete homotopy group functors we require. All proofs can be found in [25].

Let X be a finite, nonempty set and let θ be a binary reflexive relation on X. We say that (X, θ) is a *structure* or a *digraph*. For notational ease, and when there is no possible confusion, we shall talk about the structure X and omit reference to the relation θ. Thinking of the structures as digraphs, we write $x \to y$ if $(x, y) \in \theta$. A *homomorphism* from a structure X to Y is a map $f : X \longrightarrow Y$ such that $f(x) \to f(x')$ in Y whenever $x \to x'$ in X. A *pointed structure* is a pair (X, x_0) where X is a structure and $x_0 \in X$. A homomorphism from (X, x_0) to (Y, y_0) is a base-point preserving homomorphism $f : X \longrightarrow Y$, i.e. such that $f(x_0) = y_0$.

The product of structures is the usual one: we have $(x, y) \to (x', y')$ in the product structure $X \times Y$ precisely when $x \to x'$ and $y \to y'$. If (X, x_0) and (Y, y_0) are pointed structures their product is the pointed structure $(X \times Y, (x_0, y_0))$. Products with more than two factors are defined in the obvious way; we shall denote the product of n copies of the structure X by X^n, and similarly for pointed structures.

Given two structures X and Y, define a structure $Hom(X,Y)$ as follows: its vertices are the homomorphisms $f : X \longrightarrow Y$ and we let $f \to g$ if $f(x) \to g(y)$ in Y whenever $x \to y$ in X. If the structures are pointed, the definition is similar. The following lemma, which states that composition of maps is structure-preserving, is immediate:

Lemma 2.1. *Let $g_1, g_2 : X \longrightarrow Y$ and let $f_1, f_2 : Y \longrightarrow Z$ be structure homomorphisms. If $f_1 \to f_2$ and $g_1 \to g_2$ then $f_1 \circ g_1 \to f_2 \circ g_2$.*

The *one-way infinite fence* F is the following structure (see Figure 1): its base set is the set of non-negative integers, and the relation is as follows: $n \to m$ if $n = m$ or n is even and $|n - m| = 1$.

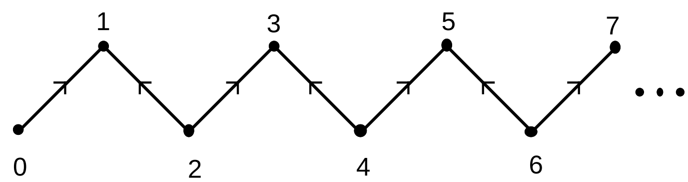

Figure 1: The one-way infinite fence F.

We say that two vertices x and y of a structure X are in the same *connected component* if there exists a homomorphism from F to X whose image contains them both. The structure X is *connected* if it has only one connected component.

Let k be a positive integer and let (X, x_0) be a pointed structure. Define the structure $F^k(X, x_0)$ as follows: its vertices are all the homomorphisms $f : F^k \longrightarrow X$ such that there exists some $N \geq 0$ with $f(t_1, \ldots, t_k) = x_0$ whenever $t_i = 0$ or $t_i \geq N$ for some index i; and the adjacency is that inherited from the structure $Hom(F^k, X)$.

Let $\sigma_k(X, x_0)$ denote the set of connected components of $F^k(X, x_0)$. For $f \in F^k(X, x_0)$, $[f]$ shall denote the connected component of $F^k(X, x_0)$ containing f. We define a group structure on $\sigma_k(X, x_0)$ as follows: let $f, g \in F^k(X, x_0)$ and let N be an even integer such that $f(t_1, \ldots, t_k) = x_0$ whenever $t_1 \geq N$. Define $(f, g)_N \in F^k(X, x_0)$ by

$$(f,g)_N(t_1, t_2, \ldots, t_k) = \begin{cases} f(t_1, t_2, \ldots, t_k) & \text{if } t_1 \leq N, \\ g(t_1 - N, t_2, \ldots, t_k) & \text{otherwise.} \end{cases}$$

Then let

$$[f] * [g] = [(f,g)_N].$$

It can be shown that this is a well-defined group operation on $\sigma_k(X, x_0)$, with neutral element $[\overline{x_0}]$, where $\overline{x_0}$ denotes the constant map with value x_0. If ϕ is a homomorphism from (X, x_0) to (Y, y_0), define the map $\phi_{\#} : \sigma_k(X, x_0) \to \sigma_k(Y, y_0)$ by $\phi_{\#}([f]) = [\phi \circ f]$ for all $f \in F^k(X, x_0)$. This is a group homomorphism, and in fact

Theorem 2.2. *The correspondence which assigns to a pointed structure* (X, x_0) *the group* $\sigma_k(X, x_0)$ *and to a homomorphism* ϕ *the map* $\phi_\#$ *defines a covariant functor from the category of pointed reflexive binary structures to the category of groups with homomorphisms.*

It can be shown that if X is connected, then the group $\sigma_k(X, x_0)$ does not depend on the choice of base-point, up to isomorphism. Thus if X is connected we often denote $\sigma_k(X, x_0)$ simply by $\sigma_k(X)$. Let X be a reflexive binary structure. We'll say that X is *0-connected* if it is connected; for $n \geq 1$ call a connected structure X *n-connected* if $\sigma_k(X) = 0$ for all $k \leq n$.

Let (X, x_0) be a pointed structure. An *n-ary operation on* X is a homomorphism $f : X^n \longrightarrow X$. If f is an operation on X we say that X *admits* f. We say that f is an operation *on* (X, x_0) (or is *base-point preserving*) if furthermore $f(x_0, \ldots, x_0) = x_0$.

We are now in a position to state our main theorem:

Theorem 2.3. *Let* X *be a finite, connected, binary reflexive structure. If* X *admits a Taylor operation, then* X *is* n*-connected for all* $n \geq 0$.

2.2 Auxiliary results on homotopy groups

Lemma 2.4. *Let* (A, a_0) *and* (B, b_0) *be two pointed structures and denote by* α *and* β *the projections of* $A \times B$ *onto* A *and* B *respectively. Then the correspondence*

$$\begin{array}{rcl} \sigma_k(A \times B, (a_0, b_0)) & \longrightarrow & \sigma_k(A, a_0) \times \sigma_k(B, b_0) \\ x & \longmapsto & (\alpha_\#(x), \beta_\#(x)) \end{array}$$

is a group isomorphism.

Proof. Simply notice that the correspondence

$$\begin{array}{rcl} F^k(A \times B, (a_0, b_0)) & \longrightarrow & F^k(A, a_0) \times F^k(B, b_0) \\ h & \longmapsto & (\alpha \circ h, \beta \circ h) \end{array}$$

is an isomorphism of structures, and that the connected components of a product are the products of the components of the factors. The rest follows easily.

We may of course generalise this last lemma to products with more than 3 factors. We shall be only interested in the following special case:

Lemma 2.5. *Let* n *be a positive integer, let* (X, x_0) *be a pointed structure, and denote by* α^i *the* $i-th$ *projection of* X^n *onto* X. *Then the correspondence*

$$\begin{array}{rcl} \sigma_k(X^n, (x_0, \ldots, x_0)) & \longrightarrow & \sigma_k(X, x_0)^n \\ x & \longmapsto & (\alpha^1_\#(x), \ldots, \alpha^n_\#(x)) \end{array}$$

is a group isomorphism.

By the last lemma, there is an isomorphism ψ_n from the group $\sigma_k(X,x_0)^n$ onto $\sigma_k(X^n,(x_0,\dots,x_0))$. In particular if $f : X^n \longrightarrow X$ is an operation on (X,x_0), then the group homomorphism $f_* = f_\# \circ \psi_n$ is an n-ary operation on the set $\sigma_k(X,x_0)$. If $n=1$, we take $\psi_n = id$ so $f_* = f_\#$.

Let A and B be non-empty sets. Let $f : A^n \longrightarrow A$ and $g_i : B^k \longrightarrow A$ be functions, $1 \leq i \leq n$. We denote by $f(g_1,\dots,g_n)$ the function from B^k to A defined by

$$f(g_1,\dots,g_n)(t) = f(g_1(t),\dots,g_n(t))$$

for all $t \in B^k$.

Lemma 2.6. *Let n be a positive integer and let (X,x_0) be a pointed structure. Let f be an n-ary operation on (X,x_0) and let $g^1,\dots,g^n$ be m-ary operations on (X,x_0).*

1. *For all $h_i \in F^k(X,x_0)$, we have that*

$$f_*([h_1],\dots,[h_n]) = [f(h_1,\dots,h_n)];$$

2. *If f is a projection then so is f_*;*
3. *if f is a Taylor operation, then so is f_*;*
4. *$f(g^1,\dots,g^n)_* = f_*(g^1_*,\dots,g^n_*)$.*

Proof. (1) Define $h : F^k \longrightarrow X^n$ by $h(t) = (h_1(t),\dots,h_n(t))$. Then

$$f_*([h_1],\dots,[h_n]) = f_\# \circ \psi_n([h_1],\dots,[h_n]) = f_\#([h]) = [f \circ h] = [f(h_1,\dots,h_n)].$$

Both (2) and (3) follow easily from (1), and (4) is straightforward, with a couple of applications of (1).

Let (X,x_0) and (Y,y_0) be pointed structures. If f and g lie in the same connected component of $Hom((X,x_0),(Y,y_0))$ we say that they are *homotopic*.

Lemma 2.7. *Let (X,x_0) and (Y,y_0) be pointed structures, let f be a base-point preserving homomorphism from X to Y and let g be a base-point preserving homomorphism from Y to X such that $f \circ g$ is homotopic to the identity on Y and $g \circ f$ is homotopic to the identity on X. Then $f_\#$ is an isomorphism from $\sigma_k(X,x_0)$ onto $\sigma_k(Y,y_0)$ with inverse $g_\#$.*

Proof. First notice that if two maps ϕ and ψ are homotopic then $\phi_\# = \psi_\#$. Indeed, it is clearly sufficient to prove this in the case where $\phi \to \psi$, and then by Lemma 2.1 we have that $\phi \circ h \to \psi \circ h$ for any $h \in F^k(X,x_0)$, so $\phi_\#([h]) = [\phi \circ h] = [\psi \circ h] = \psi_\#([h])$. Now if $f \circ g$ and $g \circ f$ are homotopic to the identity, it follows that

$$f_\# \circ g_\# = (f \circ g)_\# = (id_Y)_\# = id$$

and

$$g_\# \circ f_\# = (g \circ f)_\# = (id_X)_\# = id$$

so $f_\#$ and $g_\#$ are isomorphisms and inverses of one another.

2.3 Auxiliary results on binary reflexive structures

Let X and Y be structures. A homomorphism $r : X \longrightarrow Y$ is a *retraction* if there exists a homomorphism e from Y to X such that $r \circ e$ is the identity on Y; if this holds we say that Y is a *retract of* X. Notice that the image of Y under e is an induced substructure R of X isomorphic to Y, and that $e \circ r$ is a self-map of X that fixes all elements of R.

Let H be a structure, and let $\{X_h : h \in H\}$ be a family of structures indexed by the set H. The *sum of the* X_h *over* H is the following structure: its universe is the disjoint union on the universes of the X_h; and we let $x \to y$ if either $x \to y$ in some X_h or $x \in X_a$ and $y \in X_b$ where $a \to b$ in H.

Lemma 2.8. *Let X be a finite structure. If π is an automorphism of X such that $id \to \pi$ (or equivalently $\pi \to id$) then X is the sum of the orbits of π over some structure H. Furthermore, there exists a retraction r of X onto a substructure isomorphic to H such that $r \to id$ and $id \to r$.*

Proof. Let π be an automorphism such that $id \to \pi$. By Lemma 2.1 we have that $id \to \pi^i$ for all i; in particular $id \to \pi^{-1}$, and composing each side by π we obtain that $\pi \to id$. In fact, it is clear that we obtain that $\pi^i \to \pi^j$ for all i, j. Consequently, every orbit of π is a complete, symmetric graph, i.e. $\pi^i(x) \to \pi^j(x)$ for all i, j and all $x \in X$.[1] Now let $x \to y$ in X; it follows that $\pi^i(x) \to \pi^j(y)$ for all i, j. Choose representatives $h_1, \dots, h_m$ of each orbit of π and let $H = \{h_1, \dots, h_m\}$ viewed as an induced substructure of X. It is clear that X is the sum of the orbits of π over this structure H.

Define the retraction in the obvious way, mapping an element $x \in X$ to the representative h_i in its orbit under π. It is immediate that r is a homomorphism, and that $r \to id$ and $id \to r$.

Remark. (Not needed in sequel) It is easy to see that, conversely, if X is a sum of complete symmetric graphs X_h over a structure H, then any permutation π of X whose orbits are precisely the X_h is an automorphism of X with $id \to \pi$ and $\pi \to id$.

Lemma 2.9. *Let X be a finite, connected, binary reflexive structure such that the identity is alone in its connected component of $Hom(X, X)$. Let f be an $n+1$-ary idempotent operation on X which is not a projection. Then for every $a_1, \dots, a_n \in X$, the map $x \mapsto f(a_1, \dots, a_n, x)$ is not onto.*

Proof. Consider the map $g : X^n \longrightarrow Hom(X, X)$ where $g(a_1, \dots, a_n)(x) = f(a_1, \dots, a_n, x)$ for all $a_i, x \in X$. Clearly this is a homomorphism. Since X is connected so is the image of g. Suppose there exists some $a_i \in X$ such that $\pi = g(a_1, \dots, a_n)$ is onto. Let $\pi \to F$ (the case $F \to \pi$ is similar.) Then by Lemma 2.1 we have that $\pi \circ \pi^{-1} \to F \circ \pi^{-1}$ so $id = F \circ \pi^{-1}$ whence $F = \pi$. Thus π is the only element in the image of g and therefore f does not depend on

[1] Alternatively, the finite binary reflexive structure $Aut(X, X)$ admits a group operation (the composition) and hence it admits a Mal'tsev operation. It is well-known that the only reflexive binary relations invariant under a Mal'tsev operation are equivalence relations (see [29].)

its first n variables, and since it is idempotent, we get that f is the projection on the last variable, a contradiction.

Finally, we require a result which follows immediately from Corollary 1.2 (2) of [32]:

Lemma 2.10. *Let X be a finite, binary reflexive structure. If there exists some $n \geq 2$ such that every idempotent n-ary operation on X is a projection, then the same holds for all $n \geq 2$.*

2.4 Proof of Theorem 2.3.

We shall now state and prove a result from which our main theorem will follow. We say that a structure X is *idempotent trivial* if the only idempotent operations on X are the projections.

Theorem 2.11. *Let X be a finite, connected, binary reflexive structure. Suppose there exists a $k \geq 1$ such that*

1. *$\sigma_k(X)$ is a non-trivial group, and*
2. *$\sigma_k(R) = 0$ for every proper retract R of X.*

Then X is idempotent trivial.

Proof. Let $x_0 \in X$, and let $\sigma = \sigma_k(X, x_0)$ be non-trivial, such that $\sigma_k(R) = 0$ for every proper retract R of X. In the following it will be convenient to denote the group σ additively, i.e. its group operation will be denoted by $+$ and its neutral element by 0, even though the group might not be Abelian. Let id_σ denote the identity endomorphism of σ and let 0 denote the constant endomorphism.

Claim 1. The identity is alone in its connected component of $Hom(X, X)$.

Proof of Claim 1. Case 1: suppose that there is a non-onto homomorphism $f : X \longrightarrow X$ such that $f \rightarrow id$ (the case $id \rightarrow f$ is identical.) By Lemma 2.1 we obtain that $f^n \rightarrow id$ for all n; hence we may assume that f is a retraction onto some proper retract R of X. Let g denote the embedding of R in X. Clearly $g \circ f = f \rightarrow id$ and $f \circ g$ is the identity on R. Let u_0 be any element of R. Then the pointed spaces (X, u_0) and (R, u_0) and the maps f and g satisfy the conditions of Lemma 2.7. But $\sigma_k(R) = 0$ and σ is non-trivial, so this case is untenable. Case 2: now suppose that $\pi \rightarrow id$ for some automorphism π. By Lemma 2.8, X is the sum of the orbits of π over some structure H and there exists a retraction r of X onto H that satisfies $r \rightarrow id$. Invoking Lemma 2.7 as in the first case, we obtain that the group σ is isomorphic to $\sigma_k(H, u_0)$ for some $u_0 \in H$; it follows that $H = X$, so every orbit of π has one element, i.e. $\pi = id$.

Suppose for a contradiction that X is not idempotent trivial. By Lemma 2.10, there must exist some non-trivial binary idempotent operation f on X. For any $u, v \in \sigma$ we have that

$$\begin{aligned} f_*(u, v) &= f_*(u, 0) + f_*(0, v) \\ &= \lambda(u) + \rho(v) \end{aligned}$$

for some endomorphisms λ, ρ of σ.

Claim 2. $\lambda(u) + \rho(u) = u$ for all $u \in \sigma$.

Proof of Claim 2. Let $[\gamma] \in \sigma_k(X, x_0)$. Since f is idempotent we have by Lemma 2.6 (1) that

$$f_*([\gamma], [\gamma]) = [f(\gamma, \gamma)] = [\gamma]$$

hence f_* is also idempotent. It follows that for any $u \in \sigma$ we have

$$\lambda(u) + \rho(u) = f_*(u, u) = u$$

and the claim follows.

Now consider the following self-maps of X: let $L(x) = f(x, x_0)$ and let $R(x) = f(x_0, x)$ for all $x \in X$.

Claim 3. $L_* = \lambda$ and $R_* = \rho$.

Proof of Claim 3. If ι stands for the identity map on X, and $\overline{x_0}$ stands for the unary constant map with value x_0 on X, then clearly $\iota_* = id_\sigma$ and $\overline{x_0}_* = 0$. It follows from Lemma 2.6 (4) that for every $u \in \sigma$ we have

$$\begin{aligned} L_*(u) &= f(\iota, \overline{x_0})_*(u) \\ &= f_*(\iota_*, \overline{x_0}_*)(u) \\ &= f_*(u, 0) \\ &= \lambda(u). \end{aligned}$$

The proof for ρ is identical.

Claim 4. There exist positive integers p and q such that $\lambda^p = 0$ and $\rho^q = 0$ (where λ^p denotes the composition of λ with itself p times).

Proof of Claim 4. By symmetry it suffices to prove the claim for λ. By Claim 1 and Lemma 2.9 the map L is not onto, and thus there exists some positive integer p such that L^p is a retraction onto a proper retract R of X. For any $[\gamma] \in \sigma_k(X, x_0)$ $L^p \circ \gamma$ lies in $F^k(R, x_0)$, which is connected since $\sigma_k(R)$ is trivial. Thus there is a path in $F^k(X, x_0)$ from $L^p \circ \gamma$ to the constant map $\overline{x_0}$, and hence $(L^p)_\#([\gamma]) = [L^p \circ \gamma] = [\overline{x_0}]$. By Claim 3, and since $L_* = L_\#$, we then have that

$$(\lambda)^p = (L_\#)^p = (L^p)_\# = \overline{x_0}_\# = 0.$$

We shall now define a non-trivial Abelian subgroup τ of σ which is invariant under f_*. Fix any non-trivial element $s \in \sigma$ and let $C(s)$ denote the set of all elements t of σ such that $s + t = t + s$. It is a subgroup of σ, and it is easy to see that because f_* is idempotent it preserves $C(s)$, i.e. $f_*(t_1, t_2) \in C(s)$ for all $t_1, t_2 \in C(s)$. Let τ be the center of $C(s)$, i.e. the set of all elements t of $C(s)$ such that $t + t' = t' + t$ for all $t' \in C(s)$. Since τ contains s it is non-trivial; it is obviously Abelian, and it is easy to verify that f_* preserves it. In particular, the restrictions of λ and ρ to τ are elements of the ring $End(\tau)$ of endomorphisms of τ; for simplicity we denote the restrictions also by λ and ρ. As usual, we let 1 denote the identity endomorphism and 0 the constant endomorphism.

Now it follows from Claims 2 and 4 that

$$\lambda + \rho = 1,$$
$$\lambda^p = 0$$
$$\rho^q = 0$$

hold in $End(\tau)$. In particular, ρ is an invertible element of $End(\tau)$: indeed, we have that

$$\rho(1+\lambda+\lambda^2+\cdots+\lambda^{p-1}) = (1-\lambda)(1+\lambda+\lambda^2+\cdots+\lambda^{p-1}) = 1-\lambda^p = 1.$$

But then ρ is both nilpotent and invertible, which means that $0 = 1$ in $End(\tau)$, a contradiction since τ is non-trivial. □

Proof of Theorem 2.3. Suppose for a contradiction that for some $k \geq 1$, $\sigma_k(X)$ is not trivial for some connected, finite, reflexive binary structure X admitting a Taylor term. Let X be a counterexample of minimal cardinality. We claim that if R is a proper retract of X then $\sigma_k(R)$ is trivial. Indeed, it is easy to see that, if a structure X admits a Taylor term f, then every retract of X admits a Taylor term: if r is a retraction of X onto R and f is an n-ary Taylor term on X, then the restriction of $r \circ f$ to R^n is the desired operation. Since X is connected, so is R, and the claim follows by induction hypothesis. Thus the previous result applies and we conclude that X is idempotent trivial, a contradiction. □

3 Applications

3.1 Computing the groups $\sigma_k(X)$

To apply Theorem 2.3 we need methods to determine whether some group $\sigma_k(X)$ of a structure X is trivial or not. To each structure X we associate a simplicial complex as follows. Let $\underline{n}$ denote the n-element chain, i.e. the poset on $\{0, 1, 2, \ldots, n-1\}$ with the usual ordering. A *simplex in* X is a subset Y of X which is the homomorphic image of a chain. Alternatively, a subset Y of X is a simplex if the restriction of the edge relation to Y^2 contains a transitive tournament. The collection of simplices of X forms a simplicial complex: we denote its geometric realisation by $\widehat{X}$. We say that X *triangulates* a topological space T if $\widehat{X}$ is homeomorphic to T.

The next result follows from Theorem 5.7 of [25], and states that the group $\sigma_k(X, x_0)$ is isomorphic to the k-th homotopy group of the associated (pointed) topological space:

Theorem 3.1. *For every pointed reflexive structure* (X, x_0)*, the group* $\sigma_k(X, x_0)$ *is isomorphic to the group* $\pi_k(\widehat{X}, \widehat{\{x_0\}})$.

3.2 Idempotent trivial structures

In 1985, while investigating the fixed-point property for finite posets, E. Corominas conjectured that *if a finite connected poset P admits a fixed-point free order-preserving self-map but none of its proper retracts does, then P is idempotent trivial* (see [6]). Although various special cases have been proved, the general conjecture remains open. We note in passing the following special case we can deduce immediately from Theorem 2.11. It follows from Theorem 3.1 and a result of Baclawski and Björner [1] that if a finite connected poset P has a fixed-point free order-preserving self-map then $\sigma_k(P)$ must be non-trivial for some $k \geq 1$; hence if every proper retract R of P has $\sigma_k(R) = 0$, then P is idempotent trivial by Theorem 2.11. Hence:

Theorem 3.2. *Let P be a finite connected poset with a fixed-point free order-preserving self-map such that every proper retract of P has trivial homotopy. Then P is idempotent trivial.*

Idempotent trivial structures have been studied by various authors; in particular, the following posets have been shown to be idempotent trivial: (a) crowns ([6] and [11]), (b) truncated Boolean lattices [6], (c) braids of reach greater than 2 [12], (d) ramified posets of height 1 [17], (e) ordinal sums of non-trivial ramified posets [23]; and various reflexive binary structures such as cycles of length 5 or more [32] (see 3.3.1 below). References [10] and [17] contain other interesting examples of idempotent trivial structures.

Theorem 2.11 generalises several of the above results and provides a large class of new examples. In particular, crowns, truncated Boolean lattices, the directed cycle of length 3 and cycles of length at least 4 with any given orientation are all structures that triangulate spheres.

Theorem 3.3. *Let X be a structure that triangulates a sphere of dimension $k \geq 1$. Then X is idempotent trivial.*

Proof. Since the sphere $\mathbb{S}^k$ is path-connected, X is connected. Since $\pi_k(\mathbb{S}^k) = \mathbb{Z}$, $\sigma_k(X)$ is a non-trivial group by Theorem 3.1. To apply Theorem 2.11 it remains to show that $\sigma_k(R) = 0$ for every proper retract of X. But if R is a proper retract of X, then $\widehat{R}$ is a proper retract of $\widehat{X}$, by functoriality (see the remarks following Theorem 4.11 and Corollary 5.6 in [25]). It follows that $\widehat{R}$ is a retract of the sphere minus a point, which is a contractible space, and thus $\pi_k(\widehat{R}) = 0$. Another application of Theorem 3.1 gives the result. □

As a small sample of new examples, consider the structure on $\{0, 1, \ldots, n\}$ with all possible edges *except* $0 \to n \to n-1 \to \cdots \to 2 \to 1 \to 0$, $n \geq 2$, see Figure 2. It is easy to see that every subset is a simplex except X itself; it follows that X triangulates the sphere $\mathbb{S}^{n-1}$, and we conclude from our last result that X is idempotent trivial. We note in passing that these structures have the following interesting characterisation: they are minimal with the property that they are not a simplex, but all their subsets are. Since these structures admit no non-trivial idempotent operations, the associated retraction problem is **NP**-complete by Theorem 1.1; in the following we give several other structures with this property.

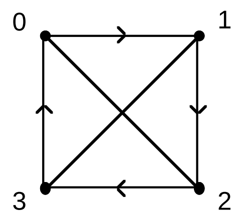

Figure 2: A minimal "non-simplex" on 4 vertices.

3.3 NP-completeness results

We now present various applications of Theorem 2.3 in combination with Theorem 1.1. Let X be a binary reflexive relational structure, with binary relation θ. We let $CSP(X)$ denote the problem $CSP(\Gamma)$ where

$$\Gamma = \{\theta\} \bigcup \{\{x\} : x \in X\},$$

i.e. $CSP(X)$ is the retraction problem for X, as discussed in the introduction. By Theorem 1.1 we know that $CSP(X)$ is **NP**-complete if X admits no Taylor operation, and is conjecturally in **P** otherwise. Furthermore, as pointed out earlier, the full dichotomy conjecture follows from a dichotomy for the problems $CSP(X)$.

We show first that there is no loss of generality in considering only connected structures.

Proposition 3.4. *Let X be a finite, binary reflexive structure with connected components $X_1, \ldots, X_s$.*

1. *If each $CSP(X_i)$ is in* **P** *then $CSP(X)$ is in* **P***;*
2. *If there is some i such that $CSP(X_i)$ is* **NP***-complete then so is $CSP(X)$;*
3. *X admits a Taylor operation if and only if each X_i admits a Taylor operation.*

Proof. (1) This is a simple reduction: let G be an input to the problem $CSP(X)$, i.e. it is a digraph with certain vertices constrained to be mapped to fixed vertices of H (i.e. "coloured" vertices). If there is some connected component of G that contains vertices coloured by vertices from two distinct components of X then there is obviously no homomorphism from G to X. Hence, each component of G has vertices coloured from at most one component X_i, and thus to determine if G maps to X amounts to determining whether each component maps to the corresponding component of X.

(2) We show that $CSP(X_i)$ reduces to $CSP(X)$. Let G be an input to $CSP(X_i)$; since G maps to X_i if and only if each component of G does, we may safely assume that G is connected. If no vertex is coloured then any constant map is a homomorphism from G to X_i, so we may assume that at least one vertex is coloured, by some vertex in X_i. But then G maps to X_i if and only if it maps to X.

(3) If t is an n-ary idempotent operation on X then it must preserve each component: indeed, since X_i^n is connected, so is its image under t, and since t is idempotent it must map X_i^n to X_i. Hence X_i is invariant under any idempotent operation on X, in particular X_i admits a Taylor operation.

To show the converse, it clearly suffices to show that if a structure X is the disjoint union of structures X_1 and X_2 that admit Taylor operations t_1 and t_2 respectively, then X admits a Taylor operation. For $i = 1, 2$ let t_i be m_i-ary, and assume without loss of generality (simply by adding a fictitious variable if necessary) that both x and y appear on both sides of each of the Taylor identities for t_1 and t_2: furthermore, let $p_i \in X_{3-i}$ be some fixed elements. Define idempotent operations s_i (of arity m_i) on X as follows: s_i is equal to t_i on $X_i^{m_i}$, is the first projection on $X_{3-i}^{m_i}$, and is equal to p_i on all other tuples. It is easy to verify that s_i is a homomorphism and is idempotent, for $i = 1, 2$. Consider the $m_1 m_2$-ary operation on X defined by

$$s(x_{1,1}, \ldots, x_{m_1,m_2}) = s_1(s_2(x_{1,1}, \ldots, x_{1,m_2}), \ldots, s_2(x_{m_1,1}, \ldots, x_{m_1,m_2})).$$

We claim that this is a Taylor operation. Indeed, if

$$t_1(x_1^1, \ldots, x_{m_1}^1) \approx t_1(x_1^2, \ldots, x_{m_1}^2)$$

denotes the i-th Taylor identity for t_1 and

$$t_2(u_1^1, \ldots, u_{m_2}^1) \approx t_2(u_1^2, \ldots, u_{m_2}^2)$$

denotes the j-th Taylor identity for t_2, then we have that s satisfies the following Taylor identity for the variable $x_{i,j}$:

$$s(z_{1,1}^1, \ldots, z_{m_1,m_2}^1) \approx s(z_{1,1}^2, \ldots, z_{m_1,m_2}^2)$$

where, for every a, b, k we have

$$z_{a,b}^k = \begin{cases} u_b^2 & \text{if } x_a^k = y, \\ u_b^1 & \text{if } x_a^k = x. \end{cases}$$

The proof, although a bit tedious, is quite straightforward, and is left to the reader (note that the condition that both x and y appear on both sides of all Taylor identities for t_1 and t_2 is required in the case where the values for x and y are in different components of X.) □

Combining Theorems 1.1 and 2.3 and the last result yields the following general hardness criterion:

Theorem 3.5. *Let Γ be a set of relations on A containing all singleton unary relations, and let θ be a binary, reflexive relation inferred from Γ. If the structure $X = \langle A; \theta \rangle$ has a non-trivial homotopy group, then the problem $CSP(\Gamma)$ is* **NP***-complete.*

3.3.1 Intransitive digraphs

We'll say that a structure X is *intransitive* if the following condition holds: for all $x, y, z \in X$, if $x \to y \to z$ and $x \to z$ then $\{x, y, z\}$ contains at most 2 elements. If X is a structure, its *symmetric closure* is the structure X' with edge relation θ that satisfies $(x, y) \in \theta$ if and only if $x \to y$ or $y \to x$ in X. For every $n \geq 3$ let C_n denote the structure with base set $\{0, 1, \ldots, n-1\}$ and relation θ defined by $(i, j) \in \theta$ if $|i - j| \leq 1$ modulo n. We'll say a structure X is a *cycle* if its symmetric closure is isomorphic to some C_n. Obviously cycles on 4 vertices or more are intransitive, as is the 3-cycle of Figure 3. In fact, any digraph whose symmetric closure has girth at least 4 is intransitive.

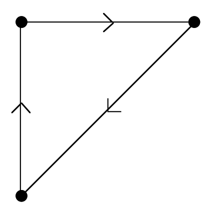

Figure 3: The intransitive 3-cycle.

We'll say that a structure X is a *forest* if it contains no cycle, and a *tree* is a connected forest. A 3-ary operation m on a set A is a *majority* operation if it satisfies the identities

$$m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x.$$

Theorem 3.6. *Let X be an intransitive structure. Then the following conditions are equivalent:*

1. *X admits a Taylor operation;*
2. *X admits a majority operation;*
3. *X is a forest.*

If X satisfies one of the above conditions then the problem $CSP(X)$ is in **P***, and it is* **NP***-complete otherwise.*

Proof. The second statement will follow from the first using Theorem 1.1 and the fact that structures invariant under a majority operation are known to have a tractable retraction problem [20]. Now we prove the equivalence: if X is intransitive, then it contains no simplex of size greater than 2, and hence the simplicial complex associated to X has dimension (at most) 1. It is well-known that each component of such a complex has a trivial fundamental group if and only if it is a tree, thus by Theorems 2.3 and 3.1 if an intransitive structure admits a Taylor operation it is a forest. Since a majority operation is a Taylor operation, it remains to show that every forest admits a majority operation. In [13] it is shown that every tree admits a majority operation: simply define $m(x_1, x_2, x_3)$ to be the unique vertex that lies on all the paths joining x_i to x_j, $1 \leq i \neq j \leq 3$. Now we show that if the components X_i of a structure X each admit a majority operation m_i then X admits a majority operation. Indeed, define an operation m on X as follows: $m(x_1, x_2, x_3) = m_i(x_1, x_2, x_3)$ if all the x_j belong to X_i; $m(x_1, x_2, x_3) = x_1$ if the x_i are all in distinct components; and $m(x_1, x_2, x_3) = x_j$ if precisely two entries x_j and x_k are in the same component, where $j < k$. It is easy to verify that m is a majority operation and that it is a homomorphism. □

3.3.2 Tournaments

We'll say that a structure X is a *tournament* if the following condition holds: for every distinct vertices $x, y \in X$, exactly one of $x \to y$ or $y \to x$ holds.

Theorem 3.7. *Let X be a tournament. Then X admits a Taylor operation if and only if it is transitive. If X is transitive, then the problem $CSP(X)$ is in* **P***, and it is* **NP***-complete otherwise.*

Proof. It is easy to see that a tournament X is transitive if and only if it is *acyclic*, i.e. does not contain a directed cycle. The acyclic tournament is actually a linearly ordered poset and thus admits various nicely behaved Taylor operations such as min and max, which are semilattice operations, and also a majority operation. Structures admitting such terms are known to have a tractable retraction problem [20].

It thus remains to show that a non-acyclic tournament X does not admit a Taylor operation. Suppose for a contradiction that there is a counterexample; choose one with the smallest number of vertices. Since X is a non-acyclic tournament it contains an intransitive 3-cycle $a \to b \to c \to a$. Let $T = \{a, b, c\}$. Since T triangulates a sphere and X admits a Taylor operation, X must contain T properly.

Claim. There exists $u \in X \setminus T$ and $\alpha, \beta \in T$ such that $\alpha \to u \to \beta$.

Proof of Claim. Consider the following construction: let Y be the set of all $y \in X$ such that there exist $z_i \in X$ with

$$a \to z_1 \to b \to z_2 \to c \to z_3 \to a$$

and $z_i \to y$ for all $1 \leq i \leq 3$; and let Z consist of all $z \in X$ such that there exist $w_i \in X$ with

$$a \to w_1 \to b \to w_2 \to c \to w_3 \to a$$

and $z \to w_i$ for all $1 \leq i \leq 3$. It is easy to see that $T \subseteq Y \cap Z$ and thus $Y \cap Z$ is not acyclic; by minimality of X we conclude that $X = Y \cap Z$.

Suppose that no element satisfies the conditions of the claim. Then for every $x \in X$, the elements w_i and z_i above are in T. It is easy to see that in this case there are at least two distinct z_i's and similarly for the w_i's. Since they are all in T, we conclude that there exists some $t \in T$ such that $t \to x \to t$ and hence $X = T$, a contradiction.

Without loss of generality we may assume that there is a vertex $u \in X \setminus T$ such that $u \to t$ for precisely two vertices $t \in T$ (otherwise we could consider the structure obtained from X by reversing all edges), and by symmetry we may assume these are a and b. Consider the set V that consists of all vertices $v \in X$ such that $a \to s \to b$ and $s \to v$ for some $s \in X$. Clearly V contains T and so by minimality we have $V = X$. It follows that we can find a vertex d such that $a \to d \to b$ and $d \to u$: we have the situation depicted in Figure 4. Notice that such a d cannot be in $\{a, b, c, u\}$. There are now two cases to consider: (i) first suppose that $d \to c$, and consider the set of all y such that $d \to y$. It contains the 3-cycle $\{b, c, u\}$, but it does not contain a, contradicting the minimality of X. (ii) Now suppose that $c \to d$, and consider the set of all z such that $c \to z$. It contains the 3-cycle $\{u, a, d\}$ but not b, contradicting the minimality of X. □

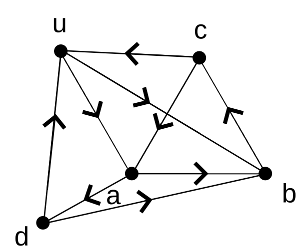

Figure 4: A configuration in X.

3.3.3 Series-parallel structures

Recall from 3.1 that the 2-element chain is the 2-element poset on $\{0, 1\}$ with the usual ordering. A structure is *series-parallel* if it can be constructed from copies of the one-element structure by disjoint unions and sums over the 2-element chain (as defined in section 2.3). It is easy to see that any such structure is antisymmetric and transitive, i.e. a poset. We'll say that a structure Y is an *inferred substructure* of the structure X if it is an induced substructure of X

whose universe is inferred from X as a unary relation (i.e. is invariant under every idempotent operation on X.)

For $n \geq 2$, an n-ary operation f on A is a *totally symmetric idempotent operation (TSI)* if it is idempotent and satisfies the identity

$$f(x_1, \ldots, x_n) \approx f(y_1, \ldots, y_n)$$

whenever $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}$. These Taylor operations generalise in many ways semilattice operations; in particular it is known that a structure admitting a TSI operation has a tractable retraction problem, see [9] and [16]. The following result is proved for connected posets in [8] (see also [21]), and relies on a combinatorial description of the connected series-parallel posets admitting Taylor operations. Its extension to non-connected structures is straightforward:

Theorem 3.8. *Let X be a series-parallel structure. Then the following conditions are equivalent:*

1. *X admits a Taylor operation;*
2. *X admits TSI operations of every arity $n \geq 2$;*
3. *$\sigma_1(Y) = 0$ for every connected, inferred substructure of X.*

If X satisfies one of the above conditions then $CSP(X)$ is in **P**, *otherwise it is* **NP***-complete.*

Proof. The implications (2) $\Rightarrow$ (1) $\Rightarrow$ (3) are obvious. Now suppose that (3) holds. Since idempotent operations on X preserve connected components, it follows that inferred substructures of a component are also inferred substructures of X, and hence (3) holds for every component. It follows that every component has TSI operations of every arity. Thus it remains to show the following: if a structure X is the disjoint union of structures X_1 and X_2 that admit n-ary TSI operations f_1 and f_2 respectively, then X admits an n-ary TSI operation. Define the operation f as follows: let u be any vertex of X. Let $f(x_1, \ldots, x_n) = f_i(x_1, \ldots, x_n)$ if the x_j are all in X_i, and let $f(x_1, \ldots, x_n) = u$ otherwise. It is clear that f is a homomorphism and satisfies the TSI identities. $\square$

Notice that we could have added condition (3) above to the equivalent statements in Theorem 3.6. Furthermore, close inspection of the proof of Theorem 3.7 shows that every non-acyclic tournament admits a directed 3-cycle as an inferred substructure, and thus condition (3) is also equivalent to the existence of a Taylor operation on tournaments. It is clear that for any structure X admitting a Taylor operation, the homotopy groups of every inferred substructure must be trivial. It would be interesting to find an example of a structure that satisfies this last condition but does not admit a Taylor operation (see [22] for related results on posets).

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