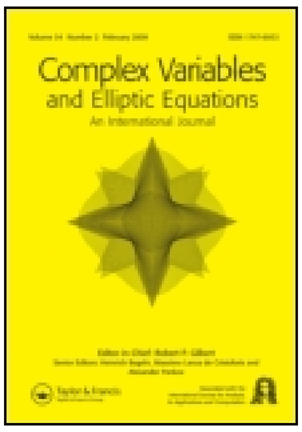

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Badih Ghusayni [a]

[a] Department of Mathematics, North Central College, Naperville, Illinois, 60566

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Integral Representation of 2π-Periodic and Trigonometrically Convex Functions

BADIH GHUSAYNI

Department of Mathematics, North Central College, Naperville, Illinois 60566

The integral representation given by Levin [1, p. 60, Theorem 24] of 2π-periodic and ρ-trigonometrically convex functions which are indicators of holomorphic functions of non-zero finite order ρ is incorrect. Counterexamples are given here as well as a corrected version of the representation.

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We need the following lemma whose proof follows easily from the definition of trigonometric convexity:

LEMMA (a) *If $h(y)$ and $k(y)$ are ρ-trigonometrically convex functions and c is a non-negative real number, then $h(y)+k(y)$ and $ch(y)$ are ρ-trigonometrically convex.*

(b) *For any real number A, $A\cos\rho y$ and $A\sin\rho y$ are ρ-trigonometrically convex.*

(c) *Let $\{h_n(x)\}$ be a sequence of ρ-trigonometrically convex functions such that $h(y)=\lim h_n(y)$, as $n\to\infty$, exists. Then $h(y)$ is ρ-trigonometrically convex.*

Following is a counterexample for Levin's theorem [1, p. 60, Theorem 24] for the case of non-integral ρ and another counterexample for the case of integral ρ: Define $h(y)$ on $[0, 2\pi)$ by:

$$h(y) = -\cos\frac{2(y-\pi)}{3} + \frac{2\sqrt{3}}{3}\sin\frac{y}{2}$$

and extend it periodically with period 2π. Then, if

$$s(x) = h'(x) + \rho^2\int^x h(t)\,dt,$$

with $\rho = \frac{2}{3}$ we get

$$s(x) = -\tfrac{7}{27}\sqrt{3}\cos\frac{x}{2}.$$

Now since $s'(x) = \frac{7}{54}\sqrt{3}\sin\dfrac{x}{2} \geqslant 0$ on $[0, 2\pi)$, it follows that $s(x)$ is a nondecreasing function. Thus, from Levin [1, p. 57], we conclude that $h(y)$ is ρ-trigonometrically convex. However,

$$\begin{aligned}\frac{1}{2\rho\sin\pi\rho'}\int_{y-2\pi}^{y}\cos\rho(y-x-\pi)\,ds(x) &= \frac{7}{36}\int_{y-2\pi}^{y}\cos\frac{2(y-x-\pi)}{3}\sin\frac{x}{2}\,dx\\ &= \frac{7}{72}\int_{y-2\pi}^{y}\left\{\sin\left(\frac{2y}{3}-\frac{x}{6}-\frac{2\pi}{3}\right)\right.\\ &\qquad\left.+\sin\left(\frac{2y}{3}-\frac{7x}{6}-\frac{2\pi}{3}\right)\right\}dx\\ &= -\tfrac{13}{24}\cos\frac{y}{2}+\frac{\sqrt{3}}{24}\sin\frac{y}{2},\end{aligned}$$

which is different from $h(y)$ (take for instance $y=0$) and consequently the representation given by Levin [1, p. 60, Theorem 24] is incorrect for the case of non-integral ρ.

Next, we give a counterexample for the case when ρ is an integer: Choose $s(x)=x$ which is clearly a non-decreasing function satisfying

$$\int_0^{2\pi} e^{i\rho x}\,ds(x)=0.$$

Integration by parts gives

$$\frac{1}{2\pi\rho}\int_{y-2\pi}^{y}(y-x)\sin\rho(y-x)\,ds(x)=-\frac{1}{\rho^2}.$$

By parts (a) and (b) of the lemma, the function

$$k(y)=h(y)-A\cos\rho y-B\sin\rho y$$

is trigonometrically convex. Moreover, $k(y)$ is 2π-periodic. Thus in the representation given by Levin [1, p. 60, Theorem 24], $k(y)<0$. We now show that this is impossible. From Levin [1, p. 93], there exists an entire function f of order ρ $(0<\rho<\infty)$ whose indicator coincides with $k(y)$. Using the definition of the indicator function [6] and the Maximum Principle [3, p. 229, Theorem 10.24] it is easy to see that $|f(z)|\leq 1$ for all z. Since $f(z)$ is entire and bounded, $f(z)$ must reduce to a constant by Liouville's theorem. Consequently $f(z)$ is of order $\rho=0$. This is the desired contradiction.

A corrected version of [2, p. 60, Theorem 24] is the following:

THEOREM *The general form of a 2π-periodic and ρ-trigonometrically convex function $h(y)$ is the following:*

(a) *for non-integral ρ,*

$$h(y)=\frac{1}{2\rho\sin\pi\rho}\left\{\int_0^{y}\cos\rho(y-x-\pi)\,ds(x)+\int_y^{2\pi}\cos\rho(y-x+\pi)\,ds(x)\right\};$$

(b) *for integral* ρ,

$$h(y)=\frac{1}{2\pi\rho}\int_0^{2\pi}(x-y)\sin\rho(y-x)\,ds(x)$$
$$-\frac{1}{\rho}\int_y^{2\pi}\sin\rho(y-x)\,ds(x)+A\cos\rho y+B\sin\rho y,$$

where

$$\int_0^{2\pi}e^{i\rho x}\,ds(x)=0,$$

$$A=\frac{1}{\pi}\int_0^{2\pi}h(x)\cos\rho x\,dx \qquad and \qquad B=\frac{1}{\pi}\int_0^{2\pi}h(x)\sin\rho x\,dx.$$

In both cases $s(y)$ *is given by*

$$s(y)=h'(y)+\rho^2\int^y h(t)\,dt.$$

Conversely, if $s(x)$ *is a non-decreasing function and if* $h(y)$ *is defined as in part* (a), *then* $h(y)$ *is* ρ*-trigonometrically convex. If* $s(x)$ *is a non-decreasing function satisfying*

$$\int_0^{2\pi}e^{i\rho x}\,ds(x)=0,$$

and if $h(y)$ *is defined as in part* (b) (*where* A *and* B *are arbitrary real constants*), *then* $h(y)$ *is* ρ*-trigonometrically convex.*

Proof We need to establish a one-to-one correspondence between the 2π-periodic trigonometrically convex functions $h(y)$ and the non-decreasing functions $s(y)$ satisfying the statements of the theorem.

First, suppose that $h(y)$ is a 2π-periodic trigonometrically convex function. Then $h(y)$ has a derivative at all points except possibly on a countable set N (cf. [1, p. 55]). Let

$$s(y)=h'(y)+\rho^2\int^y h(t)\,dt$$

if y is not in N, and if y is in N use either the derivative of $f(y)$ (whose indicator is $h(y)$) from the left or from the right instead of $h'(y)$, which exist by [2, p. 54]. Then, by Levin [1, p. 57], $s(y)$ is a non-decreasing function on $[0, 2\pi]$.

Secondly, we show that every non-decreasing function $s(y)$ on $[0, 2\pi]$ determines a 2π-periodic trigonometrically convex function $h(y)$. By Levin [1, p. 57], it suffices to show that there exists $h(y)$ such that

$$s(y)=h'(y)+\rho^2\int^y h(t)\,dt.$$

Equivalently, we must construct the Green's function $G(x, y)$ for the differential operator $h'' + \rho^2 h$ with some boundary conditions (we are assuming here that $s(y)$ is differentiable, with the case of non-differentiable $s(y)$ treated later in this paper).

To determine these boundary conditions observe that

$$\int_0^{2\pi} G(x, y)\{h''(x) + \rho^2 h(x)\}\, dx = \int_0^{2\pi} G(x, y) h''(x)\, dx + \rho^2 \int_0^{2\pi} G(x, y) h(x)\, dx.$$

Integrating by parts twice we can write

$$\text{(1)} \quad \int_0^{2\pi} G(x, y)\{h''(x) + \rho^2 h(x)\}\, dx = G(2\pi, y) h'(2\pi) - G(0, y) h'(0) - \{G_x(2\pi, y) h(2\pi) - G_x(0, y) h(0)\} + \int_0^{2\pi} \{G_{xx}(x, y) + \rho^2 G(x, y)\} h(x)\, dx.$$

Hence we require of G the periodic boundary conditions

$$G(0, y) = G(2\pi, y), \qquad G_x(0, y) = G_x(2\pi, y)$$

and

$$G_{xx}(x, y) + \rho^2 G(x, y) = 0.$$

Now, the general solution of

$$h''(x) + \rho^2 h(x) = 0$$

is

$$h_0(x) = A \cos \rho x + B \sin \rho x.$$

From the above boundary conditions, we can write

$$h_0(0) = h_0(2\pi) \qquad \text{and} \qquad h_0'(0) = h_0'(2\pi).$$

Thus we obtain

$$(1 - \cos 2\pi\rho) A - (\sin 2\pi\rho) B = 0, \qquad (\sin 2\pi\rho) A + (1 - \cos 2\pi\rho) B = 0.$$

The determinant of coefficients of A and B in the above system is $2(1 - \cos 2\pi\rho)$. Thus we consider two cases:

Case 1 ρ is non-integral: In this case $1 - \cos 2\pi\rho$ is non-zero. Thus $A = B = 0$ and consequently there is no non-trivial solution $h_0(x)$ of

$$h''(x) + \rho^2 h(x) = 0$$

under the prescribed boundary conditions. Since

$$G_{xx}(x, y) + \rho^2 G(x, y) = 0,$$

we have

$$G(x, y) = \begin{cases} c_1 \cos \rho x + c_2 \sin \rho x, & \text{if } 0 \leqslant x < y \\ c_3 \cos \rho x + c_4 \sin \rho x, & \text{if } y < x \leqslant 2\pi, \end{cases}$$

where c_i is a function $c_i(y)$ $(i = 1, 2, 3, 4)$. By the boundary condition

$$G(0, x) = G(2\pi, x)$$

we have

(2) $$c_1 = c_3 \cos 2\pi\rho + c_4 \sin 2\pi\rho .$$

Since $G(x, y)$ is differentiable as a function of x for a fixed y,

$$G_x(x, y) = \begin{cases} -\rho c_1 \sin \rho x + \rho c_2 \cos \rho x, & \text{if } 0 \leqslant x < y \\ -\rho c_3 \sin \rho x + \rho c_4 \cos \rho x, & \text{if } y < x \leqslant 2\pi . \end{cases}$$

By the boundary condition

$$G_x(0, y) = G_x(2\pi, y)$$

we get

(3) $$c_2 = -c_3 \sin 2\pi\rho + c_4 \cos 2\pi\rho .$$

Since

$$G_x(y+0, y) - G_x(y-0, y) = -1$$

we easily see that

(4) $$(c_1 - c_3)\rho \sin \rho y + (c_2 - c_4)(-\rho \cos \rho y) = -1 .$$

By the continuity of $G(x, y)$ at (x, y), $0 \leqslant x, y \leqslant 2\pi$,

(5) $$(c_1 - c_3) \cos \rho y + (c_2 - c_4) \sin \rho y = 0 .$$

Solving equations (3.4) and (3.5) for $c_1 - c_3$ and $c_2 - c_4$ we see that

(6) $$c_1 - c_3 = -\sin(\rho y)/\rho$$

and

(7) $$c_2 - c_4 = \cos(\rho y)/\rho .$$

Substituting

$$c_3 = c_1 + \sin(\rho y)/\rho \qquad \text{and} \qquad c_4 = c_2 - \cos(\rho y)/\rho$$

into equations (2) and (3) we easily see that

(8) $$(1 - \cos 2\pi\rho)c_1 - (\sin 2\pi\rho)c_2 = \sin \rho(y - 2\pi)/\rho ,$$

(9) $$(\sin 2\pi\rho)c_1 + (1 - \cos 2\pi\rho)c_2 = -\cos \rho(y - 2\pi)/\rho .$$

Solving equations (8) and (9) for c_1 and c_2 we find that

$$c_1 = -\cos \rho(y - \pi)/2\rho \sin \pi\rho \qquad \text{and} \qquad c_2 = -\sin \rho(y - \pi)/2\rho \sin \pi\rho .$$

Now

$$c_3 = -\cos \rho(y + \pi)/2\rho \sin \pi\rho \qquad \text{and} \qquad c_4 = -\sin \rho(y + \pi)/2\rho \sin \pi\rho .$$

Thus, it is easy to see that

$$G(x, y) = \begin{cases} -\dfrac{1}{2\rho \sin \pi\rho} \cos \rho(y - x - \pi), & \text{if } 0 \leqslant x < y \\ -\dfrac{1}{2\rho \sin \pi\rho} \cos \rho(y - x + \pi), & \text{if } y < x \leqslant 2\pi . \end{cases}$$

It is easy to check that $G(x, y)$ is a Green's function. Moreover, by Yosida [2, p. 71], $G(x, y)$ is uniquely determined.

If $s'(y)$ is continuous, then by Yosida [2, p. 66]

$$h(y) = \int_0^{2\pi} G(x, y)\{-s'(x)\}\, dx$$

and thus

$$h(y) = \frac{1}{2\rho \sin \pi\rho} \left\{ \int_0^y \cos \rho(y - x - \pi)\, ds(x) + \int_y^{2\pi} \cos \rho(y - x + \pi)\, ds(x) \right\}.$$

If $s'(x)$ is not continuous, then since $s\left(\frac{y}{2\pi}\right)$ is continuous on $[0, 2\pi]$ it follows by the Bernstein theorem (cf. [5, p. 5]) that

$$\lim B_n(s, y/2\pi) = s(y/2\pi), \qquad \text{as} \quad n \to \infty$$

uniformly on $[0, 2\pi]$, where B_n is the Bernstein polynomial. Moreover, since $s(y/2\pi)$ is a non-decreasing function, $\{B_n(s, y/2\pi)\}$ is a sequence of non-decreasing functions [6, p. 23]. Furthermore, it is clear that $B_n(s, y/2\pi)$ is differentiable and $B_n'(s, y/2\pi)$ is continuous for each $n = 1, 2, 3, \ldots$. Thus, letting $t = y/2\pi$, there exist trigonometrically convex functions $h_n(t)$ such that

$$h_n(t) = \frac{1}{2\rho \sin \pi\rho} \left\{ \int_0^t \cos \rho(t - x - \pi)\, dB_n(s, x) + \int_t^{2\pi} \cos \rho(t - x + \pi)\, dB_n(s, x) \right\}.$$

Hence, it is clear that $h(t) = \lim h_n(t)$, as $n \to \infty$, exists. Moreover, $h(t)$ is trigonometrically convex by the lemma. Furthermore, by Rudin [13, p. 139, Theorem 7.16]

$$h(t) = \frac{1}{2\rho \sin \pi\rho} \left\{ \int_0^t \cos \rho(t - x - \pi)\, ds(x) + \int_t^{2\pi} \cos \rho(t - x + \pi)\, ds(x) \right\}.$$

If $s(y)$ is a non-decreasing but non-differentiable function, we can still approximate s by a sequence of non-decreasing Bernstein polynomials and then pass to the limit as before (cf. [6, p. 23]).

Case 2 ρ is integral: Let $s(y)$ be a non-decreasing function on $[0, 2\pi]$ satisfying

$$\int_0^{2\pi} e^{i\rho x}\, ds(x) = 0.$$

Using the method of approximation as in (a) we can assume without loss of generality that $s(y)$ is differentiable and $s'(y)$ is continuous on $[0, 2\pi]$. Hence consider

$$h''(y) + \rho^2 h(y) = s'(y).$$

If we proceed as in the proof of (a) using equation (1) to require of $G(x, y)$ that $G_{xx}(x, y) + \rho^2 G(x, y) = 0$, $G(0, y) = G(2\pi, y)$ and $G_x(0, y) = G_x(2\pi, y)$, we notice that a solution $h(y)$ of the equation

$$s(y) = h'(y) + \rho^2 \int^y h(t)\, dt$$

cannot be constructed when ρ is an integer, because in this case the homogeneous differential equation

$$h''(y)+\rho^2 h(y)=0$$

has the periodic solutions $\cos \rho y$ and $\sin \rho y$ and therefore there is no Green's function satisfying the periodic boundary conditions. To see this, we argue by contradiction. Suppose

$$G(x,y)=\begin{cases} c_1 \cos \rho x + c_2 \sin \rho x, & \text{if } 0 \leqslant x < y \\ c_3 \cos \rho x + c_4 \sin \rho x, & \text{if } y < x \leqslant 2\pi, \end{cases}$$

where c_i is a function $c_i(y)$ $(i=1,2,3,4)$.

By the boundary conditions

$$G(0,y)=G(2\pi,y), \qquad G_x(0,y)=G_x(2\pi,y),$$

it follows that $c_1=c_3$ and $c_2=c_4$. Now

$$G_x(y+0,y)-G_x(y-0,y)=-1$$

gives the desired contradiction. Nevertheless, we can construct a generalized Green's function that gives the periodic solution $h(y)$ of the inhomogeneous differential equation

$$h''(y)+\rho^2 h(y)=s'(y)$$

when $s'(y)$ is orthogonal to the solutions $\cos \rho y$ and $\sin \rho y$ (i.e. $\int_0^{2\pi} e^{i\rho x} s'(x)\,dx=0$) of the homogeneous differential equation by requiring that

$$G_{xx}(x,y)+\rho^2 G(x,y)=-\delta(x-y)+\alpha_1 \cos \rho y \cos \rho x+\alpha_2 \sin \rho y \sin \rho x$$

with α_1 and α_2 chosen so that

$$\int_0^{2\pi} (\sin \rho x)\{-\delta(x-y)+\alpha_1 \cos \rho y \cos \rho x+\alpha_2 \sin \rho y \sin \rho x\}\,dx=0$$

and

$$\int_0^{2\pi} (\cos \rho x)\{-\delta(x-y)+\alpha_1 \cos \rho y \cos \rho x+\alpha_2 \sin \rho y \sin \rho x\}\,dx=0,$$

where $\delta(t)$ is Dirac's δ-function ($\delta(t)=0$ if t is different from zero). Using the facts that

$$\int_0^{2\pi} (\sin \rho x)\,\delta(x-y)\,dx=\sin \rho y \qquad \text{and} \qquad \int_0^{2\pi} (\cos \rho x)\,\delta(x-y)\,dx=\cos \rho y,$$

a direct computation shows that $\alpha_1=\alpha_2=1/\pi$. Thus the required condition becomes

$$G_{xx}(x,y)+\rho^2 G(x,y)=-\delta(x-y)+\frac{1}{\pi}\cos \rho(y-x).$$

Since $\delta(x-y)=0$ when x is different from y and since $-(1/2\pi\rho)\{x \sin \rho(y-x)\}$ is the particular solution due to the additional $(1/\pi)\cos \rho(y-x)$ in the differential equation, we can write

$$G(x, y) = -\frac{x \sin \rho(y-x)}{2\pi\rho} + \begin{cases} A \cos \rho x + B \sin \rho x, & 0 \leqslant x < y \\ C \cos \rho x + D \sin \rho x, & y < x \leqslant 2\pi, \end{cases}$$

where A, B, C and D are functions of y to be determined. Using the boundary conditions, we obtain

(10) $$C - A = \frac{\sin \rho y}{\rho}$$

and

(11) $$B - D = \frac{\cos \rho y}{\rho}.$$

Thus

$$G = -\frac{x \sin \rho(x-y)}{2\pi\rho} + \begin{cases} A \cos \rho x + B \sin \rho x, & 0 \leqslant x < y \\ \{A \cos \rho x + B \sin \rho x\} + \dfrac{\sin \rho(y-x)}{\rho}, & y < x \leqslant 2\pi \end{cases}$$

That is

(12) $$G = -\frac{x \sin \rho(y-x)}{2\pi\rho} + A \cos \rho x + B \sin \rho x + \begin{cases} 0, & 0 \leqslant x < y \\ \dfrac{\sin \rho(y-x)}{\rho}, & y < x \leqslant 2\pi. \end{cases}$$

Since $\sin \rho x$ and $\cos \rho x$ are non-trivial solutions of

$$G_{xx}(x, y) + \rho^2 G(x, y) = 0,$$

using one condition of the definition of the generalized Green's function, we can write

(13) $$\int_0^{2\pi} G(x, y) \cos \rho x \, dx = 0$$

and

(14) $$\int_0^{2\pi} G(x, y) \sin \rho x \, dx = 0.$$

Considering equation (13), we get

$$-\frac{1}{2\pi\rho} \int_0^{2\pi} x \sin \rho(y-x) \cos \rho x \, dx + A \int_0^{2\pi} \cos^2 \rho x \, dx + B \int_0^{2\pi} \sin \rho x \cos \rho x \, dx + \frac{1}{\rho} \int_y^{2\pi} \sin \rho(y-x) \cos \rho x \, dx = 0.$$

Hence

$$-\frac{1}{4\pi\rho} \int_0^{2\pi} x\{\sin \rho y + \sin \rho(y-2x)\} \, dx + \frac{A}{2} \int_0^{2\pi} (1 + \cos 2\rho x) \, dx + \frac{B}{2} \int_0^{2\pi} \sin 2\rho x \, dx + \frac{1}{2\rho} \int_y^{2\pi} \{\sin \rho y + \sin \rho(y-2x)\} \, dx = 0.$$

But $\int_0^{2\pi} \sin 2\rho x\, dx = 0$ and hence we can solve for A by integration and get

$$A = \frac{y \sin \rho y}{2\pi\rho} + \frac{\cos \rho y}{4\pi\rho^2} - \frac{\sin \rho y}{2\rho}.$$

Similarly, by consider equation (14), we get

$$B = \frac{\sin \rho y}{4\pi\rho^2} + \frac{\cos \rho y}{2\rho} - \frac{y \cos \rho y}{2\pi\rho}.$$

Substituting A and B into equation (12) and simplifying the result we get

$$G(x, y) = \frac{(y-x)\sin \rho(y-x)}{2\pi\rho} + \frac{\cos \rho(y-x)}{4\pi\rho^2} - \frac{\sin \rho(y-x)}{2\rho} + \begin{cases} 0, & 0 \leqslant x < y \\ \dfrac{\sin \rho(y-x)}{\rho}, & y < x \leqslant 2\pi. \end{cases}$$

Now it is trivial to see that $G(x, y)$ is a generalized Green's function.

By equation (1) and the boundary conditions, we have

$$\int_0^{2\pi} G(x, y)\{h''(x) + \rho^2 h(x)\}\, dx = \int_0^{2\pi} h(x)\{G_{xx}(x, y) + \rho^2 G(x, y)\}\, dx.$$

So

$$\int_0^{2\pi} G(x, y) s'(x)\, dx = \int_0^{2\pi} h(x)\left\{-\delta(x-y) + \frac{1}{\pi}\cos \rho(y-x)\right\} dx$$

$$= -\int_0^{2\pi} h(x)\,\delta(x-y)\, dx + \frac{1}{\pi}\int_0^{2\pi} h(x)\cos \rho(y-x)\, dx$$

$$= -h(y) + \frac{1}{\pi}\int_0^{2\pi} h(x)\cos \rho(y-x)\, dx.$$

Thus

$$h(y) = -\int_0^{2\pi} G(x, y)\, ds(x) + \frac{1}{\pi}\int_0^{2\pi} h(x)\{\cos \rho y \cos \rho x + \sin \rho y \sin \rho x\}\, dx$$

$$= -\int_0^{2\pi} G(x, y)\, ds(x) + A\cos \rho y + B \sin \rho y,$$

where

$$A = \frac{1}{\pi}\int_0^{2\pi} h(x)\cos \rho x\, dx \qquad \text{and} \qquad B = \frac{1}{\pi}\int_0^{2\pi} h(x)\sin \rho x\, dx.$$

Using $\int_0^{2\pi} e^{i\rho x}\, ds(x) = 0$, a simple computation leads to

$$h(y) = \frac{1}{2\pi\rho}\int_0^{2\pi} (x-y)\sin \rho(y-x)\, ds(x)$$

$$-\frac{1}{\rho}\int_y^{2\pi} \sin \rho(y-x)\, ds(x) + A\cos \rho y + B \sin \rho y.$$

The proof of the converse follows from Levin [1, p. 57] and the Leibnitz formula [7, p. 245].

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