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On (m, n)-Anti-ideals in Ordered Semigroups

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Abstract

In this paper, we review the concept of (m, n)-anti-ideals in semigroups. This notion extends the classical concept of anti-ideals and is closely related to the study of mutants in semigroups. It is shown that a semigroup possessing a left identity admits no left (m, m)-anti-ideals, and that no (m, n)-anti-ideals exist when $m < n$. These preliminary results serve as a foundation for the subsequent development of the theory in the context of ordered semigroups.

1 Preliminaries

According to [3], Iseki introduced the concept of (m, n)-anti-ideals in semigroups. This concept is closely related to that of mutants in semigroups. Let S be a semigroup, and let A, B be non-empty subsets of S. Then A is said to be

- a *left (respectively, right)* (m, n)*-anti-ideal* of S if $SA^m \cap A^n = \emptyset$ (respectively, if $A^mS \cap A^n = \emptyset$);
- an (m, n)*-anti-ideal* of S if it is both a left and a right (m, n)-anti-ideal of S.

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Here, $AB = \{ab \mid a \in A \text{ and } b \in B\}$, $A^n = \{a_1a_2\cdots a_n \mid a_1, a_2, \ldots, a_n \in A\}$, and similarly for S^m. When $(m, n) = (1, 1)$, this reduces to the notion of an anti-ideal in the sense of Schwarz (see [4, 5]). In [3], the author proved the following results:

Theorem 1.1. *(1) Any semigroup with a left (respectively, right) identity has no left (respectively, right) (m, m)-anti-ideals.*

(2) Any semigroup with identity has no (m, m)-anti-ideals.

Proof. (1) Let S be a semigroup with a left identity e, and let A be a non-empty set of S. Then $ea = a$ for all $a \in S$, which implies $A \subseteq SA$. Therefore, $A^m \subseteq SA^m$. Thus, $SA^m \cap A^m \neq \emptyset$. Hence, S has no left (m, m)-anti-ideals.

(2) follows by (1). □

Theorem 1.2. *There are no left (respectively, right) (m, n)-anti-ideals with $m < n$ in any semigroup.*

Proof. Let S be a semigroup, and let A be a non-empty subset of S. Clearly,

$$A^2 \subseteq SA.$$

By induction, $A^{m+1} \subseteq SA^m$. Therefore, $SA^m \cap A^{m+1} \neq \emptyset$. Assume $m < n$. We then obtain the chain

$$A^n = AA^{n-1} \subseteq SA^{n-1} \subseteq S^2A^{n-2} \subseteq \ldots \subseteq S^{(n-m)}A^{n-(n-m)} \subseteq SA^m,$$

which implies $SA^m \cap A^n \neq \emptyset$. Hence, S has no left (m, n)-anti-ideals. The second part can be proved similarly. □

Remark 1.3. *There exist semigroups possessing left (or right) (m, n)-anti-ideals for $m \geq n$. Consider the additive semigroup S of positive integers, and let $A = \{1\}$. Then for $m \geq n$, we have $SA^m \cap A^n = \emptyset$.*

2 Results

In this section, we show that the results in the previous section can be extended to ordered semigroups. For ordered semigroups, we refer the reader to [1], [2].

Definition 2.1. *A semigroup S is said to be an* ordered semigroup *if there is a partial order $\leq$ on S that is compatible with the semigroup operation, that is, for all $a, b, c \in S$, $a \leq b \implies ca \leq cb$ and $ac \leq bc$.*

Example 2.2. *Let $\mathbb{Z}^+$ denote the set of all positive integers. With the usual multiplication as the binary operation and the usual order relation $\leq$, the structure $(\mathbb{Z}^+, \cdot, \leq)$ is an ordered semigroup.*

Example 2.3. *Let $\mathbb{Z}^+$ denote the set of all positive integers. With the multiplication defined by for any $a, b \in \mathbb{Z}^+$, $ab = a9b$ and the usual order relation $\leq$, the structure $(\mathbb{Z}^+, \cdot, \leq)$ is an ordered semigroup.*

Remark 2.4. *Every semigroup S can be viewed as an ordered semigroup by equipping it with the trivial partial order given by equality, that is, define $a \leq b$ if and only if $a = b$.*

Let S be an ordered semigroup, and let A, B be non-empty subsets of S. We write

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

Note that $A \subseteq (A]$, $(AB] = ((A](B]]$, and $A \subseteq B \implies (A] \subseteq (B]$.

Definition 2.5. *Let S be an ordered semigroup, and let A be a non-empty subset of S. Then A is said to be*

- *a* left (respectively, right) anti-ideal *of S if $(SA] \cap (A] = \emptyset$ (respectively, if $(AS] \cap (A] = \emptyset$);*
- *an* anti-ideal *of S if A is both a left and a right anti-ideal of S.*

Example 2.6. The ordered semigroup $(\mathbb{Z}^+, \cdot, \leq)$ in Example 2.2 has no left anti-ideals.

Example 2.7. For the ordered semigroup $(\mathbb{Z}^+, \cdot, \leq)$ in Example 2.3, we have that $\{1\}$, $\{1, 2\}$, $\{1, 2, 3\}$, $\{1, 2, 3, 4, 5\}$ are left anti-ideals.

Example 2.8. Let $S = \{a, b, c, d, e\}$ be an ordered semigroup with Cayley table and Hasse diagram as follows:

$\cdot$	a	b	c	d	e
a	a	b	a	a	a
b	a	b	a	a	a
c	a	b	a	a	a
d	a	b	a	a	a
e	a	b	a	a	e

$$\leq = \{(a, a), (a, b), (a, e), (b, b), (c, c), (c, b), (d, d), (d, b), (d, e), (e, e)\}.$$

Since $S\{d\} = \{a\}$, $(S\{d\}] = \{a\}$. By $(d] = \{d\}$, $(S\{d\}] \cap (d] = \emptyset$. Then $\{d\}$ is a left anti-ideal. From $\{d\}S = \{a, b\}$, $(\{d\}S] = \{a, b, c, d\}$. Then $\{d\}$ is not a right anti-ideal because $(\{d\}S] \cap \{d\} \neq \emptyset$. In the same manner we have that $\{c, d\}$ is a left anti-ideal, where as it is not a right anti-ideal.

Remark 2.9. *Suppose S is a semigroup. As Remark 2.4, S forms an ordered semigroup with the trivial partial order given by equality. If A is a left anti-ideal of the semigroup S; then $(SA] \cap (A] = SA \cap A = \emptyset$. Hence, A is a left anti-ideal of the ordered semigroup S.*

Definition 2.10. *Let S be an ordered semigroup. An element $e \in S$ is called a* left (respectively, right) identity *of S if $ea = a$ (respectively, if $ae = a$) for all $a \in S$. An element $e \in S$ is called an* identity *of S if $ea = a = ae$ for all $a \in S$.*

Like Theorem 1.1, the following holds:

Theorem 2.11. *(1) Any ordered semigroup with a left (respectively, right) identity has no left (respectively, right) anti-ideals.*

(2) Any ordered semigroup with identity has no anti-ideals.

Proof. Let S be an ordered semigroup with a left identity e (that is, $ex = x$ for all $x \in S$). Suppose $A \subseteq S$ is a non-empty subset. Taking any $x \in (A]$; then $x \leq a$ for some $a \in A$. Since $x = ex \leq ea \in SA$, $x \in (SA]$, which implies $(A] \subseteq (SA]$. Hence, S has no left anti-ideals. This proves (1).

(2) follows by (1). □

Definition 2.12. *Let S be an ordered semigroup, and let A be a non-empty subset of S. Then A is said to be*

- *a* left (respectively, right) (m, n)-anti-ideal *of S if $(S^m A^n] \cap (A^n] = \emptyset$ (respectively, if $(A^n S^m] \cap (A^n] = \emptyset$);*
- *an* (m, n)-anti-ideal *of S if A is both a left and a right (m, n)-anti-ideal of S.*

Analogous to Theorem 1.2, the following holds:

Theorem 2.13. *If $m < n$, then the following hold.*

(1) Any ordered semigroup has no left (respectively, right) (m, n)-anti-ideals.

(2) Any ordered semigroup with identity has no (m, n)-anti-ideals.

Proof. (1) Let S be a semigroup, and let A be a non-empty subset of S. We have

$$(SA] \supseteq (A^2].$$

By induction, we prove $(SA^m] \supseteq (A^{m+1}]$. Therefore, $(SA^m] \cap (A^{m+1}] \neq \emptyset$. Let $n > m$. We then obtain the chain

$$(SA^m] \supseteq (S^{(n-m)}A^{n-(n-m)}] \supseteq \cdots \supseteq (S^2A^{n-2}] \supseteq (SA^{n-1}] \supseteq (AA^{n-1}] = (A^n],$$

which implies $(SA^m] \cap (A^n] \neq \emptyset$. Hence, there exist no left (m, n)-anti-ideals in S.

(2) follows by (1). □

3 Conclusion

In this paper, we have extended the concept of (m, n)-anti-ideals in semigroups to the setting of ordered semigroups. It was shown that ordered semigroups with left or right identities possess no corresponding anti-ideals and that no (m, n)-anti-ideals exist when $m < n$. These results generalize known properties of semigroups and provide a basis for further investigation of anti-ideal structures in ordered algebraic systems.

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