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# Non-existence of Solution of the Diophantine Equation $19^x + 18^y = z^2$

**William S. Gayo, Jr., Ronica M. de Leon, Regan R. Rola**

Don Mariano Marcos Memorial State University
La Union, Philippines

email: wgayo@dmmmsu.edu.ph, rmasangkay@dmmmsu.edu.ph,
rrola@dmmmsu.edu.ph

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**Abstract**

In this paper, we show the non-existence of non-negative integer solution of the Diophantine equation $19^x + 18^y = z^2$.

## 1 Introduction

The non-existence of solutions is common to many Diophantine equations. For instance, the linear Diophantine equation $ax + by = c$ has no solution if the greatest common divisor of $a$ and $b$ does not divide $c$. Also, the equation $x^n + y^n = z^n$ has no non-trivial integer solution $(x, y, z)$ if $n > 2$. Moreover, the exponential Diophantine equation $p^x + (p-1)^y = z^2$, where $p$ is a prime, has no non-negative integer solution when $p = 3, 11, 17$ [1, 2, 4]. In this paper, we will show that $19^x + 18^y = z^2$ has no integer solution.

## 2 Preliminaries

Here, we present some definitions, lemmas, and theorems.

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**Theorem 2.1 (Mihailescu's Theorem).** *[3] The quadruple* $(3, 2, 2, 3)$ *is the unique solution for the Diophantine equation* $a^x - b^y = 1$ *where* $a, b, x$ *and* $y$ *are integers with* $min\{a, b, x, y\} > 1$

As corollaries to Mihailescu's Theorem, we have the following two lemmas:

**Lemma 2.2.** *[4] The triple* $(3, 1, 2)$ *is the unique non-negative integer solution of the Diophantine equation* $p^x + 1 = z^2$ *where* $p$ *is an odd prime.*

**Lemma 2.3.** *[2] The triple* $(3, 3, 3)$ *is the unique non-negative integer solution for the Diophantine equation* $1 + (p - 1)^y = z^2$ *where* $p$ *is an odd prime.*

**Definition 2.4.** *A Pythagorean triple is a set of three integers* $a, b, c$ *such that* $x^2 + y^2 = z^2$*; the triple is said to be primitive if* $\gcd(x, y, z) = 1$.

**Theorem 2.5.** *The positive integers* $x, y, z$ *form a primitive Pythagorean triple with* $y$ *even if and only if there are relatively prime positive integers* $u$ *and* $v$*,* $u > v$*, with* $u \not\equiv v (mod\ 2)$ *such that* $x = 2uv, y = u^2 - v^2$ *and* $z = u^2 + v^2$.

**Lemma 2.6.** *[2] The square of an odd integer is congruent to* $1 \pmod 8$.

**Lemma 2.7.** *[5] The square of an integer is congruent to* $0, 1, 4, 5, 6, 7, 9, 11, 16$ *or* $17 \pmod{19}$.

# 3 Main Results

**Theorem 3.1.** *The Diophantine equation* $19^x + 18^y = z^2$ *has no non-negative integer solution.*

*Proof.* Suppose there are non-negative integers $x, y$, and $z$ such that $19^x + 18^y = z^2$.

First, we study the case when $x$ or $y$ is zero. If $x = 0$, then the Diophantine equation $1 + 18^b = z^2$ has no non-negative solutions by Lemma 2.2. If $y = 0$, then the Diophantine equation $19^x + 1 = z^2$ has no integer solution also by Lemma 2.3.

Now, we consider the case when $x, y \geq 1$. Note that $19^x \equiv 3 \pmod 8$ for odd $x$, and $19^x \equiv 1 \pmod 8$ for even $x$. Moreover, $18^y \equiv 2 \pmod 8$ if $y = 1$, $18^y \equiv 4 \pmod 8$ if $y = 2$, and $18^y \equiv 0 \pmod 8$ if $y \geq 3$. Since $z^2 \equiv 1 \pmod 8$ by Lemma 2.6, it follows that $x$ is even and $y \geq 3$.

Furthermore, for every positive even integer $x$, $19^x \equiv 0 \pmod{19}$. Also, $18^y \equiv -1 \pmod{19}$ for odd $y$ and $18^y \equiv 1 \pmod{19}$for even $y$. Since $z^2 \equiv 0, 1, 4, 5, 6, 7, 9, 11, 16, 17 \pmod{19}$ by Lemma 2.7, it immediately follows that $y$ is even.

Since $x$ and $y \geq 3$ are both even, then there exist integers $m \in \mathbb{N}$ and $n \in \mathbb{N}\backslash\{1\}$ such that $19^{2m}+18^{2n} = z^2$. Observe that the triple $(19^m, 18^n, z)$ forms a primitive Pythagorean triple. Thus, by Theorem 2.5, there exist relatively prime integers $u$ and $v$ with $u > v$ such that $19^m = u^2 - v^2, 18^n = 2uv$ and $z = u^2 + v^2$. Obviously, $18^n = 2^n 3^{2n}$. Hence, $uv = 2^{n-1}3^{2n}$. Considering that $u$ and $v$ are relatively prime, we get two possibilities for $u$ and $v$.

Case 1: If $u = 3^{2n}$ and $v = 2^{n-1}$, then $19^m = 3^{4n} - 2^{2(n-1)}$. Note that $3^{4n} \equiv 3 \pmod{6}$ and $2^{2(n-1)} \equiv 4 \pmod{6}$. So $3^{4n} - 2^{2(n-1)} \equiv 5 \pmod{6}$. This is a contradiction because $19^m \equiv 1 \pmod{6}$

Case 2: If $u = 3^{2n}2^{n-1}$ and $v = 1$, then $19^n = 3^{4n}2^{2(n-1)} - 1$. Note that $3^{4n}2^{2(n-1)} \equiv 0 \pmod{6}$. So $3^{4n}2^{2(n-1)} - 1 \equiv 5 \pmod{6}$. This is a contradiction because $19^n \equiv 1 \pmod{6}$.

Hence, the equation $19^x + 18^y = z^2$ has no non-negative integer solution. □

# 4 Conclusion

We have shown that the Diophantine equation $19^x + 18^y = z^2$ has no solution in non-negative integers using a modular arithmetic method, Mihailescu's theorem, and properties of primitive Pythagorean triple.

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