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Energy of Bipartite graphs generated by Hankel sequences

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Abstract

The Hankel matrix corresponding to the sequence $(a_1, a_2, ..., a_{2n-1})$ is a matrix of order n. This matrix is graphical only when $a_1 = a_3 = a_5 = ... = a_{2n-1} = 0$ and $a_2, a_4, a_6, ...a_{2n-2}$ is either 0 or 1. The phrase "Hankel graph"refers to such a graph. The Hankel graphs so generated are the bipartite graphs. In this paper, the energy bound for bipartite graphs generated by some Hankel sequences has been determined.

1 Introduction

A graph G(V,E) is a connected, finite, simple graph with $|V(G)| = n$ and $|E(G)| = m$. Any terminology used but not defined here may be found in [1]. If G is a bipartite graph with n vertices, then the energy $E(G) \leq \frac{n}{\sqrt{8}}(\sqrt{2} + \sqrt{n})$ was discussed in ([2],[3]).

Definition 1.1. *The adjacency matrix of a graph G with n vertices and no parallel edges is an n by n symmetric $(0, 1)$ matrix is defined as*

$$A(G) = (a_{ij}) = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

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Definition 1.2. *The Hankel matrix of order n generated by the sequence $(a_1, a_2, a_3, ..., a_{2n-1})$ is graphical only when $a_1 = a_3 = a_5 = ... = a_{2n-1} = 0$ and $a_2, a_4, a_6, ...a_{2n-2}$ is either 0 or 1. The graph is known as "Hankel graph".*

Definition 1.3. *Let G be a graph with n vertices and m edges. Then the energy of a graph is defined as the sum of the absolute value of its eigenvalues $\lambda_1, \lambda_2, ...\lambda_n$; that is, $E(G) = \sum_{i=1}^{n} |\lambda_i|$.*

2 Main results

In this section, the energy of a Hankel graph generated by few sequences is discussed.

Theorem 2.1. *If G is a Hankel graph generated by the sequence $(a_1, a_2.....a_{2n-1}) = (01010.....010)$ when n is odd, then $E(G) = 2\sqrt{\lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil}$.*

Proof. *The Hankel matrix corresponding to the sequence $(a_1, a_2.....a_{2n-1}) = (01010.....010)$ when n is odd is*

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & ..... & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & ..... & 0 & 1 \\ . & & & & & & & \\ 0 & 1 & 0 & 1 & 0 & ..... & 1 & 0 \end{bmatrix}$$

The degree sequence of a graph G is $(d_1, d_2.....d_n) = (\lfloor\frac{n}{2}\rfloor, \lceil\frac{n}{2}\rceil.....\lceil\frac{n}{2}\rceil, \lfloor\frac{n}{2}\rfloor)$. In $A(G)$, $\lceil\frac{n}{2}\rceil$ number of rows are identical and having $\lfloor\frac{n}{2}\rfloor$ number of 1's in each row and $\lfloor\frac{n}{2}\rfloor$ number of rows are identical and having $\lceil\frac{n}{2}\rceil$ number of 1's in each row. Using Row reduced Echelon form, Rank of $A(G) = 2$. By the Rank-Nullity theorem, the number of non zero eigenvalues of $A(G)$ is 2 and the number of zero eigenvalues of $A(G)$ is $n-2$. The characteristic polynomial of $A(G)$ is $\lambda^{n-2}(\lambda^2$-$\lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil) = 0$. Therefore, the eigenvalues of $A(G)$ are $\lambda = 0$ repeated n-2 times, $\lambda = \pm\sqrt{\lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil}$. Hence, $E(G) = 2\sqrt{\lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil}$.

Theorem 2.2. *If G is a Hankel graph generated by the sequence $(a_1, a_2.....a_{2n-1}) = (0101.....010)$ when n is even, then $E(G) = n$.*

Proof. *The Hankel matrix with the sequence is $(a_1, a_2.....a_{2n-1}) = (01010.....010)$*

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 1 & ..... & 0 & 1 \\ 1 & 0 & 1 & 0 & ..... & 1 & 0 \\ . & & & & & & \\ 1 & 0 & 1 & 0 & ..... & 1 & 0 \end{bmatrix}$$

In $A(G)$, $\frac{n}{2}$ number of rows are identical and each row contains $\frac{n}{2}$ number of 1's. Therefore, $deg_G V = \frac{n}{2}, \forall V$. So, G is regular with regularity $\frac{n}{2}$. The number of distinct rows of $A(G)$ is 2 and hence, by reducing to Echelon form, rank of $A(G)$ = 2. By the Rank-Nullity theorem, the number of non zero eigenvalues of $A(G)$ is 2 and the number of zero eigenvalues $A(G)$ is $n-2$. If G is regular with regularity k, then one of the largest eigenvalues of $A(G)$ is k of order n. Then $A(G)$ has an eigenvalue $\lambda_1 = \frac{n}{2}$. The sum of the eigenvalues of $A(G)$ is equal to the trace of A. Thus, the trace of A is $\frac{n}{2} + \lambda_2$ and so $\lambda_2 = -\frac{n}{2}$. The eigenvalues of $A(G)$ are $\pm\frac{n}{2}$, 0 (repeated $n-2$ times. The characteristic equation is $\lambda^{n-2}(\lambda^2$-$(\frac{n}{2})^2)$ = 0. Hence, $E(G) = n$.

The proofs of the following theorems are similar.

Theorem 2.3. *If G is a Hankel graph generated by the sequence $((a_1, a_2, .., a_{2n-1}) =$ (0001010.....10) when n is odd, having m edges and degree sequence $d_1 = \lfloor\frac{n}{2}\rfloor, d_2 = \lfloor\frac{n}{2}\rfloor - 1$, then $E(G) = 2\sqrt{\left\{\frac{m}{2} \pm \frac{\sqrt{m^2 - 4d_1d_2}}{2}\right\}}$.*

Theorem 2.4. *If G is a Hankel graph generated by the sequence $(a_1, a_2, .., a_{2n-1}) =$ (00010.....10) when n is even having m edges and degree sequence $d_1 = \lfloor\frac{n}{2}\rfloor, d_2 = \lfloor\frac{n}{2}\rfloor - 1$, then $E(G) = 2\left(\sqrt{\{\frac{m}{2} + \frac{\sqrt{m^2 - 4d_1d_2}}{2}\}} + \sqrt{\{\frac{m}{2} - \frac{\sqrt{m^2 - 4d_1d_2}}{2}\}}\right)$.*

Theorem 2.5. *If G is a Hankel graph generated by the sequence $(a_1, a_2, ..., a_{2n-1}) =$ (010001010.....10) when $n = 2k-1, k = 4, 5..$ has m edges, then $E(G) = 2\sqrt{\frac{(m-1)\pm\sqrt{(m-1)^2-4[(k-3)^2+(k-3)]}}{2}}$.*

Theorem 2.6. *If G is a Hankel graph with the sequence $(a_1, a_2..., a_{2n-1}) =$ (010001010.....1010) when $n = 2k, k = 3, 4.5..$ has m edges, then $E(G) = 2\sqrt{\frac{(m-1)\pm\sqrt{(m-1)^2-4[(k^2+4-4k)]}}{2}}$.*

Theorem 2.7. *If G is a Hankel graph with the sequence $(a_1, a_2, ., a_{n+1}, a_{n+2}, ., a_{2n-1}) =$ (01...10001...1010) when n is odd, then $E(G) = 2(\sqrt{(\lfloor\frac{n}{2}\rfloor)(\lfloor\frac{n}{2}\rfloor - 1) + 1} + (\lfloor\frac{n}{2}\rfloor - 1))$.*

Theorem 2.8. *If G is a Hankel graph with the sequence $(a_1a_2, ., a_{n+1}a_{n+2}, ., a_{2n-1}) =$ (0101...100010...10) when n is even and $(d_1, ...d_n) = [\frac{n}{2}, \frac{n}{2}, \frac{n}{2} - 1, ...\frac{n}{2} - 1]$, then $E(G) = 2\lfloor\frac{n-3}{2}\rfloor + 2\sqrt{\frac{\sum d_r + \sqrt{(\sum d_r)^2 - 4}}{2}} + 2\sqrt{\frac{\sum d_r - \sqrt{(\sum d_r)^2 - 4}}{2}}$.*

Theorem 2.9. *If G is a Hankel graph with sequence* $(a_1, a_2, ., a_{n+1}, a_{n+2}, ., a_{2n-1}) = (01...010001..010)$ *having degree sequence* $\sum d_i = (\lfloor \frac{n}{2} + 1 \rfloor)(\lceil \frac{n}{2} \rceil) + (\lfloor \frac{n}{2} \rfloor - 1)^2$ *when n is odd, then* $E(G) = 2\{\lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor - \frac{\sum d_i}{n}\}$.

Theorem 2.10. *If G is a Hankel graph with the sequence* $(a_1, ., a_{n+1}, a_{n+2}, ., a_{2n-1}) = (0101...100010...10)$ *when n is even, then* $E(G) = (\frac{2n^2 - 7n + 6}{n})$.

The above mentioned Hankel graphs, which are produced by Hankel sequences, are bipartite graphs with a vertex set divided into two halves, one of which has all vertices with odd suffixes and the other part of which has even suffixes. The energy bounds for Sequence 1: 0101...10, Sequence 2: 0001...10, Sequence 3: 010001...10, Sequence 4: 010...10001...10, Sequence 5: 0101...000...10 and $E(G) \leq \frac{n}{\sqrt{8}}(\sqrt{2} + \sqrt{n})$ is done using the Sage software. Figure 1 shows the number of vertices versus the energy for various sequences and that the energy of the bipartite graph lies between n and $2n - 5$.

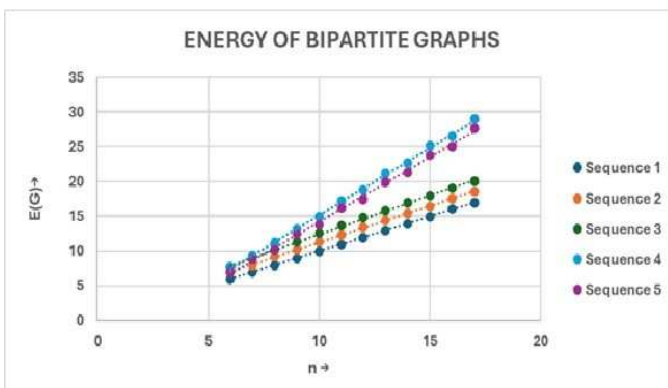

Figure 1: Energy of bipartite graphs

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