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Extending Chained Hesitant Fuzzy R-Modules and Evaluating Their Results

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Abstract

In this paper, we study chained H.F. R-modules, and define multiplication H.F. R-modules, and extend their definitions to the theory of fuzzy modules. Also key properties, examples and the importance of counteractive rings in connection with comparability, decomposability and direct sums would be discussed here. The results may also throw light on and avenues for continuing further research in algebra.

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1 Introduction

This research extends the concept of chained F. modules and HF.ideals within the framework of chained HF.R-modules. The article is divided into three sections. The first part gives the key definitions and basic results necessary for clarity and coherence in subsequent analysis. The second part investigates the properties of chained HF.R-modules, focusing on their comparability and decomposability. The last section covers the direct sum of those modules and HF.homomorphisms. The work aims at a deeper theoretical understanding of chained HF.R-modules and emphasizes their applicability in both pure and applied mathematics.

1.1 Basic Definitions

These definitions are needed now and in the future to understand this topic.

Definition 1.1. *[1] Let us consider a non-empty set $\mathcal{X}$. A hesitant fuzzy set (H.F.) A in the universe $\mathcal{X}$ is described as the function $H_A(x)$ which gives us a restricted subset of $[0,1]$ when evaluated in this space. For convenience, consider $A = (x, H_A(x))|x \in \mathcal{X})$. However, it is now clear that this function assigns many things in the region $[0,1]$, which indicates that a membership is possible for that element x in that set A. This is referred as $H_A(x)$ as an attribute of fuzzy hesitancy.*

Definition 1.2. *[1] Let $\mathcal{W}$ be an R-module, then a fuzzy set $\mathcal{X}$ of $\mathcal{W}$ is said to be a module F on an R-module $\mathcal{W}$ if the following conditions hold:*
1) $\mathcal{X}$ (n - m) $\geq$ min { $\mathcal{X}$ (n), $\mathcal{X}$ (m)}, $\forall$ n, m $\in \mathcal{W}$.
2) (rn) $\geq \mathcal{X}$ (n), for $\forall$ n $\in \mathcal{W}$ and r $\in$ R.
3) $\mathcal{X}$ $(0) = 1$, (0 denotes the zero component of $\mathcal{W}$).

Definition 1.3. *[2] Assume H to be an F. set $\mathcal{X}$, and $\mathcal{Y}$ are two H.F. sets of H, then it is called $\mathcal{X} \subseteq \mathcal{Y}$ if $\mathcal{X}(n) \subseteq \mathcal{Y}$ (n) for all $n \in H$.*

Definition 1.4. *[2] Assume H that a set of H.F. in the R-module $\mathcal{W}$ then H is called a H.F. module over the R-module $\mathcal{W}$ if $\forall$ n, m $\in \mathcal{W}$, r $\in$ R.*
(1) $H\,(n-m) \supseteq H\,(n) \cap H\,(m)$
(2) $H(rn) \supseteq \ H(n)$

Definition 1.5. *[3] Let us assume, first of all, that H is an F module of the R-module $\mathcal{W}$. We say that H is a Chained F.R-module if for any F submodule of H, $\mathcal{X}$, it can be said that either $\mathcal{X} \subseteq \mathcal{Y}$ or $\mathcal{Y} \subseteq \mathcal{X}$.*

Definition 1.6. *[4] Let H be an H.F. module and an R-module $\mathcal{W}$. Then, H is referred to as a H.F. Chained R-module in the event that each pair of H.F. submodules $\mathcal{X}$ and $\mathcal{Y}$ of H, one of the following conditions is satisfied: $\mathcal{X} \subseteq \mathcal{Y}$ or $\mathcal{Y} \subseteq \mathcal{X}$.*

Definition 1.7. *[3] Let $\mathcal{X}$ be an F subset of A for every $t \in [0,1]$, the set $\mathcal{X}_t = \{n \in A, \mathcal{X}(n) \geq t\}$ is named the scale sub-set of $\mathcal{X}$ at level t.*

Lemma 1.8. *[3] Suppose f that a function is defined in a set $\mathcal{W}$. $\mathcal{X}_1$ and $\mathcal{X}_2$ are fuzzy subsets of $\mathcal{W}$, $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are fuzzy subsets of $\mathrm{f}(\mathcal{W})$. Thus, the following statements are true:*
1) $\mathcal{X}_1 = \mathrm{f}^{-1}(\mathrm{f}(\mathcal{X}_1))$, whenever $\mathcal{X}_1$ is an f-invariant.
2) $\mathrm{f}(\mathrm{f}^{-1}(\mathrm{Y}_1)) = \mathcal{Y}_1$.
3) If $\mathcal{X}_1 \subseteq \mathcal{X}_2$, then $\mathrm{f}(\mathcal{X}_1) \subseteq \mathrm{f}(\mathcal{X}_2)$.
4) If $\mathcal{Y}_1 \subseteq \mathcal{Y}_2$, then $\mathrm{f}^{-1}(\mathcal{Y}_1) \subseteq \mathrm{f}^{-1}(\mathcal{Y}_2)$.

Definition 1.9. *[3] Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two fuzzy R-modules of an R-modules $\mathcal{W}_1$ and $\mathcal{W}_2$ in a row, $\mathrm{f} : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is known as a fuzzy homomorphism if $\mathrm{f} : \mathcal{W}_1 \longrightarrow \mathcal{W}_2$ is R-homomorphism and $\mathcal{C}_2(\mathrm{f}(n)) = \mathcal{C}_1(\mathrm{n})$ for each $\alpha \in \mathcal{W}_1$.*

Remark 1.10. *[3] The following level subset properties are true $\forall t \in [0,1]$:*
1. $(\mathcal{X} \cap \mathcal{Y})_t = \mathcal{X}_t \cap \mathcal{Y}_t$.
2. $\mathcal{X} = \mathcal{Y}$ if and only if $\mathcal{X}_t = \mathcal{Y}_t$, $\forall t \in [0,1]$.

2 New Insights into Chained H.F. R-Modules

Recall that each two submodules $\mathcal{X}$ and $\mathcal{Y}$ of a R-module $\mathcal{W}$ are considered comparable if $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{Y} \subseteq \mathcal{X}$. Additionally, if $H$is a fuzzy R-module that has two fuzzy sub-modules $\mathcal{X}$ and $\mathcal{Y}$ of an R-module $\mathcal{W}$, then $\mathcal{X}$ and $\mathcal{Y}$ are known to be comparable if $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{Y} \subseteq \mathcal{X}$ [3].
We will extend this concept by fuzzifying it as follows.

Proposition 2.1. *If each of the two H.F. submodules of H is comparable, then the H.F. module C of a R-module $\mathcal{W}$ is the Chained H.F.R-module.*

Proof. Let $\mathcal{X}$ and $\mathcal{Y}$ be H.F. submodules of C. Suppose $\mathcal{X} \nsubseteq \mathcal{Y}$, we show that $\mathcal{Y} \subseteq \mathrm{X}$ since $\mathcal{X} \nsubseteq \mathcal{Y}$, there exist $c_t \in \mathcal{X}$ and $c_t \notin \mathcal{Y}$, $< c_t > \subseteq \mathcal{X}$ and $< c_t > \nsubseteq \mathcal{Y}$. Let $u_k \in \mathcal{Y}$, then $< u_k > \subseteq \mathcal{Y}$, $< c_t >, < u_k >$ are cyclic H.F. submodules of C, then $< c_t > \subseteq < u_k >$ or $< u_k > \subseteq < c_t >$. If $< c_t > \subseteq < u_k >$ implies $< c_t > \subseteq \mathcal{Y}$ (since $< u_k > \subseteq \mathcal{Y}$). Thus, $c_t \in \mathcal{Y}$ is a contradiction. If $< u_k > \subseteq < c_t >$ implies that $< u_k > \subseteq \mathcal{X}$ (since $u_k \in \mathcal{X}$). Thus $\mathcal{Y} \subseteq \mathcal{X}$ so C is a Chained H.F. R-module. The converse is obvious. □

Remark 2.2. *The Chained H.F. R-module is indecomposable.*

Proof. Suppose C is decomposable, then $C = C_1 \bigoplus C_2$ for some H.F. submodule C_1 and C_2 of C. Thus, $C_1 \cap C_2 = \{0_1\}$ is a contradiction by Proposition 2.9 [3]. $\square$

Definition 2.3. *[2] A fuzzy ring $\mathcal{C}$ of a ring R is said to be Chained fuzzy R-module if and only if for every pair of fuzzy ideals $\mathbb{I}$, $\mathbb{J}$ of C either $\mathbb{I} \subseteq \mathbb{J}$ or $\mathbb{J} \subseteq \mathbb{I}$.*

Definition 2.4. *The C is a fuzzy ring of a ring R, called Chained H.F. R-module if and only if for each H.F. ideals $\mathbb{I}$, $\mathbb{J}$ of C, either $\mathbb{I} \subseteq \mathbb{J}$ or $\mathbb{J} \subseteq \mathbb{I}$.*

Remark 2.5. *The H.F. ring C is Chained , if and only if C_t is Chained fuzzy ring for all $t \in (0, 1]$.*

Definition 2.6. *If every non-empty H.F. submodule $\mathcal{X}$ of $\mathcal{Y}$ has a fuzzy ideal $\mathbb{I}$ of R such that $\mathcal{X} = \mathbb{I}\mathcal{Y}$, then the H.F. module C of a R-module $\mathcal{W}$ is an R-multiplication H.F. module.*

Proposition 2.7. *Let us show that C is a multiplication fuzzy R-module in $\mathcal{W}$ if R is a Chained fuzzy ring. It follows that C Chained H.F.R-module.*

Proof. Consider $\mathcal{X}$ and $\mathcal{Y}$ to be H.F. submodules of C. Then, since $\mathbb{I}_t$ and $\mathbb{J}_t$ are ideals of R and R is a Chained H.F. ring, $\mathbb{I}_t \subseteq \mathbb{J}_t$ or $\mathbb{J}_t \subseteq \mathbb{I}_t$ exist. The statement $\mathbb{I} \subseteq \mathbb{J}$ or $\mathbb{J} \subseteq \mathbb{I}$ (by Remark 2.3.[3]) thus suggests that $\mathbb{I}C \subseteq \mathbb{J}C$ or $\mathbb{J}C \subseteq \mathbb{I}C$. This means that $\mathcal{X} \subseteq \mathcal{Y}$ or $\mathcal{Y} \subseteq \mathcal{X}$ exists. $\square$

3 Direct Sum of Chained H.F. R-Modules

Remark 3.1. *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two Chained H.F. module of an R-module $\mathcal{W}_1$ and $\mathcal{W}_2$, respectively, then $\mathcal{C}_1 \bigoplus \mathcal{C}_2$ not necessarily Chained H.F. R-module of an R-module $\mathcal{W}_1 \bigoplus \mathcal{W}_2$.*

Example 3.2. *Let $\mathcal{C}_1 : \mathrm{Z}_6 \longrightarrow \left\{0, \frac{1}{3}, 1\right\}$ such that:*

$$\mathcal{C}_1(\alpha) = \begin{cases} \left\{\frac{1}{3}, 1\right\} & if \quad \alpha \in < 2 > \\ \{0\} & if \ \alpha \ \notin < 2 > \end{cases}$$

Let $\mathcal{C}_2 : \mathrm{Z}_6 \longrightarrow \left\{0, \frac{1}{3}, 1\right\}$ such that:

$$\mathcal{C}_2(\alpha) = \begin{cases} \left\{\frac{1}{3}, 1\right\} & if \quad \alpha \in < 3 > \\ \{0\} & if \quad \alpha \notin < 3 > \end{cases}$$

It is clear that $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are Chained H.F. R-module of* Z_6*. Hence* $\mathcal{C}_1 \bigoplus \mathcal{C}_2$ *is not Chained H.F. R-module* $\mathrm{Z}_6 \bigoplus \mathrm{Z}_6$*. Since there exist* $\mathcal{X}$ *and* Y*, are two H.F. submodules of* $\mathcal{C}_1 \bigoplus \mathcal{C}_2$*, Where:*

$$X(\alpha,\beta) = \begin{cases} \{\frac{1}{3}, 1\} & if \quad (\alpha,\beta) \in < 2 > \bigoplus < 0 > \\ \{0\} & if \quad (\alpha,\beta) \notin < 2 > \bigoplus < 0 > \end{cases}$$

and

$$Y(\alpha,\ \beta) = \begin{cases} \{\frac{1}{3}, 1\} & if \quad (\alpha,\beta) \in < 0 > \bigoplus < 3 > \\ \{0\} & if \quad (\alpha,\beta) \notin < 0 > \bigoplus < 3 > \end{cases}$$

But $\mathrm{X}(2,0) = \frac{1}{3}, 1,\ \mathrm{Y}(2,0) = \{0\}$*, that is* $\mathrm{X} \nsubseteq \mathrm{Y}$*. Also* $\mathrm{X}(0,3) = \{0\}$*,* $\mathrm{Y}(0,3) = \{\frac{1}{3}, 1\}$*, that is* $\mathrm{Y} \nsubseteq \mathrm{X}$*. Thus* $\mathrm{X} \bigoplus \mathrm{Y}$ *is not a Chained H.F. R-module* $\mathrm{Z}_6 \bigoplus \mathrm{Z}_6$*.*

Proposition 3.3. *Let* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *be two H.F. R-modules of an R-modules* $\mathcal{W}_1$ *and* $\mathcal{W}_2$ *respectively if* $\mathcal{C}_1 \bigoplus \mathcal{C}_2$ *is Chained H.F. R-module of* $\mathcal{W}_1 \bigoplus \mathcal{W}_2$*. Then* $\mathcal{C}_1$ *is Chained H.F. R-module of* $\mathcal{W}_1$ *and* $\mathcal{C}_2$ *is Chained H.F. R-module of* $\mathcal{W}_2$*.*

Proof. Consider X and Y as H.F. submodules of $\mathcal{C}_1$, while $\hat{\mathrm{X}}$ and $\hat{\mathrm{Y}}$ are H.F. submodules of $\mathcal{C}_2$. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are Chained H.F. R-modules, $\mathrm{X} + \mathrm{Y}$ and $\hat{\mathrm{X}} + \hat{\mathrm{Y}}$ represent two H.F. submodules of $\mathcal{C}_1 \bigoplus \mathcal{C}_2$. Either $\mathrm{X} + \hat{\mathrm{X}} \subseteq \mathrm{Y} + \hat{\mathrm{Y}}$, implying $\mathrm{X} \subseteq \mathrm{Y}$ and $\hat{\mathrm{X}} \subseteq \hat{\mathrm{Y}}$, or $\mathrm{Y} + \hat{\mathrm{Y}} \subseteq \mathrm{X} + \hat{\mathrm{X}}$, leading to $\mathrm{Y} \subseteq \mathrm{X}$ and $\hat{\mathrm{Y}} \subseteq \hat{\mathrm{X}}$. Thus, $\mathrm{X} \subseteq \mathrm{Y}$ or $\mathrm{Y} \subseteq \mathrm{X}$. Consequently, $\mathcal{C}_1$ is a Chained H.F. R-module on either choice, and $\hat{\mathrm{X}} \subseteq \hat{\mathrm{Y}}$ or $\hat{\mathrm{Y}} \subseteq \hat{\mathrm{X}}$, so $\mathcal{C}_2$ is a Chained H.F. R-module. □

Theorem 3.4. *Let* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *denote H.F. R-modules of R-modules* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *respectively. Then* $\mathrm{f} : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ *should be a H.F. epimorphism. If* $\mathcal{C}_1$ *happens to be a Chained H.F. R-module, then,* $\mathcal{C}_2$ *also becomes a Chained H.F. R-module.*

Proof. Let X, Y be two H.F. R-submodules of $\mathcal{C}_2$. By lemma (1.8), $\mathrm{f}^{-1}(\mathrm{X})$, and $\mathrm{f}^{-1}(\mathrm{Y})$ are the H.F. submodules in $\mathcal{C}_1$. Obviously, $\mathcal{C}_2$ is a chained module H.F. R. So, either $\mathrm{f}^{-1}(\mathrm{X}) \subseteq \mathrm{f}^{-1}(\mathrm{Y})$ or $\mathrm{f}^{-1}(\mathrm{Y}) \subseteq \mathrm{f}^{-1}(\mathrm{X})$. Now, if $\mathrm{f}^{-1}(\mathrm{f}(\mathrm{X})) \subseteq \mathrm{f}^{-1}(\mathrm{f}(\mathrm{Y}))$, applying lemma (1.8) shows $\mathrm{f}^{-1}(\mathrm{X}) \subseteq \mathrm{f}^{-1}(\mathrm{Y})$. By similar arguments, if $\mathrm{f}^{-1}(\mathrm{Y}) \subseteq \mathrm{f}^{-1}(\mathrm{X})$, then $\mathrm{Y} \subseteq \mathrm{X}$. Hence, $\mathcal{C}_2$ is a chained H.F. R-module. □

Theorem 3.5. *Let* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *be two H.F. R-modules of an R-modules* $\mathcal{W}_1$ *and* $\mathcal{W}_2$ *respectively. Let* $\mathrm{f} : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ *be a H.F. homomorphism, and each submodule of* $\mathcal{C}_2$ *is* F*-invariant. If* $\mathcal{C}_2$ *is a Chained H.F. R-module, then* $\mathcal{C}_1$ *is a Chained H.F. R-module.*

Proof. Let X, Y are two H.F. submodules in $\mathcal{C}_1$. Hence $\mathrm{f(X)}$, $\mathrm{f(Y)}$ are H.F. submodules in $\mathcal{C}_2$ by Lemma (1.8), since $\mathcal{C}_2$ is a Chained H.F. R-module. Then either $\mathrm{f(X)} \subseteq \mathrm{f(Y)}$ or $\mathrm{f(Y)} \subseteq \mathrm{f(X)}$. Now, if $\mathrm{f(X)} \subseteq \mathrm{f(Y)}$, then $\mathrm{f}^{-1}(\mathrm{f(X)}) \subseteq \mathrm{f}^{-1}(\mathrm{f(Y)})$ by Lemma (1.8). Hence $\mathrm{X} \subseteq \mathrm{Y}$ by Lemma (1.8). Similarly, if $\mathrm{f(Y)} \subseteq \mathrm{f(X)}$, then $\mathrm{Y} \subseteq \mathrm{X}$. Therefore $\mathcal{C}_1$ is a Chained H.F. R-module. □

4 Applications of Chained H.F. R-modules

4.1. Risk Assessment and Management: Chained H.F. R-modules are very useful in coping up with uncertainty in the data. Thus, these can enhance risk assessment and decision-making, especially in project management and finance. It also involves some applications in applied mathematics such as graph theory and algebra [5] and [6].
4.2. Fuzzy Algebraic Structures: H.F. R-modules are chains that extend classical modules to fuzzy contexts, allowing the possibility of analyzing algebra in cases of uncertainty or imprecision. The study of algebraic structures with fuzzy characteristics can extend the usual applications into more advanced areas [7]-[11].
4.3. Pattern recognition: It is possible that fields like signal and image processing can make a better handling of uncertainty and variability of data with the help of Chained H.F. R-modules. This development can improve the accuracy and reliability of pattern recognition algorithms [11] and [12].

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