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# On the Exponential Diophantine Equation $9(5^x) - p^y = z^4$

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**Abstract**

In this article, we determine all solutions to the equation of the form $9(5^x) - p^y = z^4$, where $p$ is a prime number and $x, y, z$ are nonnegative integers. By employing elementary techniques involving congruences, we establish that the equation admits solutions only when both $x$ and $y$ are odd and $p \equiv 29$ or $149 \pmod{240}$, or when $(x, y, z, p) = (0, 2, 0, 3)$ or $(0, 3, 1, 2)$.

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# 1 Introduction and Literature Review

Exponential Diophantine equations, especially those involving variables raised to integer powers, have long been of significant interest in number theory. A classic milestone in this area is Mihăilescu's proof of Catalan's conjecture, which states that $3^2 - 2^3 = 1$ is the only nontrivial solution to $a^x - b^y = 1$ where $\min(x, a, y, b) > 1$ [1]. This breakthrough has spurred extensive research into exponential Diophantine equations of the general form $a^x \pm b^y = c^z$, under various constraints on the parameters, such as prime bases or specific coefficient conditions. In [2], Laipaporn et al. studied the Diophantine equation $3^x + p(5^y) = z^2$, where $p$ is prime, employing congruence arguments to classify solutions depending on $p$ *mod* 24. Subsequent work [3] extended this analysis to ternary equations of the form $a^x + b^y + c^z = w^2$, analyzing over 1,300 candidate equations using elementary modular techniques. Building on this approach, Laipaporn et al. [4] investigated equations of the form $a^x \pm a^y = z^n$, revealing structural properties analogous to those found in the study of Mersenne primes. Subsequently, in [5], they nearly completed the classification of explicit solutions to the equation $p^x + q^{2y} = z^{2n}$, leaving open only the case where both exponents are odd. Most previous studies have focused on sums of two exponential terms, such as $3^x + p(5^y) = z^2$. However, in the past ten years, there has been increasing interest in similar equations involving the difference of two terms, for examples, $a^x \pm b^y = z^2$ [6] or $2^x - p^y = z^2$ [7] or $7^x - 5^y = z^2$ [8] or $a(2^x) - b(3^y) = c$ [9] or $m(p^x) + n(q^y) = r^z$ [10] or $a(x^m) + b(y^n) = c$ [11] or $4(7^x) - p^y = z^2$ [12] which pose new structural challenges and subtleties. Inspired by these recent developments, in this paper we investigate the quartic analogue, $9(5^x) - p^y = z^4$, where $p$ is a prime number. By combining parity arguments, the structure of perfect powers, modular congruences, and elementary number theory, we demonstrate that the equation only has solutions when $x$ and $y$ are both odd and $p \equiv 29$ and $149 \pmod{240}$ or when $(x, y, z, p) = (0, 2, 0, 3)$ and $(0, 3, 1, 2)$.

# 2 Main Results

## 2.1 Main Theorem

**Theorem 2.1.** *All solutions to the equation* $9(5^x) - p^y = z^4$ *where* $p$ *is a prime and* $x, y, z \geq 0$ *are described as follows:*

1. *If* $x = 0$ *or* $y = 0$, *then the only solutions are* $(x, y, z, p) = (0, 2, 0, 3)$ *and* $(0, 3, 1, 2)$.

2. *If* $x > 0$ *and* $y > 0$*, then both* $x$ *and* $y$ *are odd integers,* $z \equiv 2, 4, 8, 14, 16, 22, 26,$ *or* $28 \pmod{30}$ *and the solutions belong to* $A \cup B$*, where* $A = \{(x, y, z, p) | x \equiv y \pmod 4$ *and* $p \equiv 29 \pmod{240}\}$ *and* $B = \{(x, y, z, p) | x \not\equiv y \pmod 4$ *and* $p \equiv 149 \pmod{240}\}$.

*Proof.* It is straightforward to verify that $(y, z, p) = (2, 0, 3)$ and $(3, 1, 2)$ satisfy the equation $9 - p^y = z^4$. For $z > 1$, we observe that $p^y = 9 - z^4 < 0$, which is impossible. Thus, no other solutions exist for the equation $9 - p^y = z^4$. Reducing both sides modulo 3 gives: $9(5^x) - 1 \equiv 2 \pmod 3, z^4 \equiv 0$ or $1 \pmod 3$, which is a contradiction. Hence, we also conclude that no solution exists for the equation $9(5^x) - 1 = z^4$. It remains to consider the case $x$ and $y$ are positive. We analyze this scenario by examining the following four cases:
**Case 1:** $p = 2$. If $x$ is even, then we have $2^m = 3(5^{\frac{x}{2}}) - z^2$ and $2^n = 3(5^{\frac{x}{2}}) + z^2$ for some positive integers $m < n$ since $z$ must be odd. This gives $3(5^{\frac{x}{2}}) = 2^{m-1}(2^{n-m} + 1)$ implying that $m = 1$. Therefore, we have $z^2 = 3(5^{\frac{x}{2}}) - 2$. But modulo $5, z^2 \equiv 0, 1$ or $4 \pmod 5$, whereas $3(5^{\frac{x}{2}}) - 2 \equiv 3 \pmod 5$, a contradiction. If $x$ is odd, then we find that $9(5^x) - 2^y \equiv 5 \pmod 8$ for any $y \geq 3$, while $z^4 \equiv 1 \pmod 8$ for any positive integer $z$. This suggests that $y < 3$. However, checking $y = 1, 2$ shows that $9(5^x) - 2 \equiv 3 \not\equiv z^4 \pmod 5$ and $9(5^x) - 2^2 \equiv 11 \not\equiv z^4 \pmod{15}$, which is again a contradiction. Therefore, the equation $9(5^x) - 2^y = z^4$ has no solution.
**Case 2:** $p = 3$. Starting from the equations $9((5^x) - 3^{y-2}) = z^4$, it is clear that $3|z$. Thus, we have $\frac{z^4}{9} \equiv 0 \pmod 3$, which contradicts $5^x - 3^{y-2} \equiv 1, 2 \pmod 3$ for any $y \geq 3$. For the case $y = 1$, we adopt a similar contradiction using the same argument as in the case $y \geq 3$. Now only the case $y = 2$ remains. Consider the equation $9((5^x) - 1) = z^4$. Since $3|z$, we have $5^x - 1 \equiv \frac{z^4}{9} \equiv 0 \pmod 3$, which implies that $x$ must be even. By Catalan's conjecture, there are no pairs $(x, z)$ that satisfy the equation $5^x - (\frac{z^2}{3})^2 = 1$. Thus, we can conclude that the equation $9(5^x) - 3^y = z^4$ has no solution.
**Case 3:** $p = 5$. Consider the equation $9(5^x) - 5^y = z^4$. Since $z^4$ is non-negative, we have $x + 1 \geq y$ and if $y = x$ or $x + 1$ then a contradiction arises when the equation $9(5^x) - 5^y = z^4$ is analyzed with modulo 16 and modulo 8, respectively. For the case $x > y$, we can rewrite the equation as $5^y(9(5^{x-y}) - 1) = z^4$. Since $\gcd(5^y, 9(5^{x-y}) - 1) = 1$, it follows that $9(5^{x-y}) - 1 = t^4$ for some positive integer $t$. Applying modulo 5 to the equation $9(5^{x-y}) - 1 = t^4$ yields an immediate contradiction. Therefore, we conclude that the equation $9(5^x) - 5^y = z^4$ has no solution.
**Case 4:** $p \geq 7$. To proceed, we separate the proof into two subcases based on the parity of $x$.

*Subcase 4.1* $x$ is even. Then, we can express $p^y$ as: $p^y = (3(5^{\frac{x}{2}}) - z^2)(3(5^{\frac{x}{2}}) + z^2)$. We observe that $\gcd(z, 3) = \gcd(z, 5) = 1$, which implies $\gcd(3(5^{\frac{x}{2}}) - z^2, 3(5^{\frac{x}{2}}) + z^2) = 1$ or 2. If $\gcd(3(5^{\frac{x}{2}}) - z^2, 3(5^{\frac{x}{2}}) + z^2) = 1$, then we must have $3(5^{\frac{x}{2}}) + z^2 = p^y$ and $3(5^{\frac{x}{2}}) - z^2 = 1$. Applying modulo 3 to the equation $3(5^{\frac{x}{2}}) - z^2 = 1$ leads to a contradiction.
If $\gcd(3(5^{\frac{x}{2}}) - z^2, 3(5^{\frac{x}{2}}) + z^2) = 2$, then it follows that $p = 2$ which contradicts $p \geq 7$. Thus, we conclude that the equation $9(5^x) - p^y = z^4$ has no solution in this subcase.

*Subcase 4.2* $x$ is odd. Observe that $p \equiv 1$ or $-1 \pmod 6$ and $z$ is even. Since $z^4 \equiv 0$ or $4 \pmod 6$ and

$$9(5^x) - p^y \equiv 3(-1) - p^y \equiv \begin{cases} 4 \pmod 6 & \text{if } y \text{ is odd and } p \equiv -1 \pmod 6, \\ 2 \pmod 6, & \text{otherwise,} \end{cases}$$

it implies that $y$ must be odd and $p \equiv 5 \pmod 6$. To analyze the equation $9(5^x) - p^y = z^4$ modulo 5, note that $5 \nmid p$ and $z^4 \equiv 0$ or $1 \pmod 5$, which leads to $p^y \equiv 4 \pmod 5$. Given that $y = 4k + 1$ or $4k + 3$ for some positive integer $k$ (as $\phi(5) = 4$). It follows that $p \equiv p^{4k+1} \equiv 4 \pmod 5$ or $p^3 \equiv p^{4k+3} \equiv 4 \pmod 5$ forcing $p \equiv 4 \pmod 5$. By the Chinese Remainder Theorem, this yields $p \equiv 29 \pmod{30}$. Since both $x$ and $y$ are odd, we also find that $z^4 \equiv 9(5) - (-1)^y \equiv 16 \pmod{30}$ which restricts $z$ to $\pm 2$ or $\pm 4$ or $\pm 8$ or $\pm 14 \pmod{30}$. Moreover, since $z \equiv \pm 2$ or $\pm 4$ or $\pm 8$ or $\pm 14 \pmod{30}$, we have $z^4 \equiv 0 \pmod 8$. Also, note that $9(5^x) \equiv \begin{cases} 5 \pmod 8 & \text{if } x \text{ is odd,} \\ 1 \pmod 8 & \text{if } x \text{ is even,} \end{cases}$ and $p^y \equiv \begin{cases} p \pmod 8, & \text{if } y \text{ is odd,} \\ 1 \pmod 8, & \text{if } y \text{ is even.} \end{cases}$ Applying the equation $9(5^x) - p^y = z^4$ modulo 8 and using the fact that both $x$ and $y$ are odd, it follows that $p \equiv 5 \pmod 8$. Moreover, applying the Chinese Remainder Theorem once more to $p \equiv 5 \pmod 8$ and $p \equiv 29 \pmod{30}$, we obtain $p \equiv 29 \pmod{120}$. Furthermore, every solution $(x, y, z, p)$ lies in $S = \{(2i + 1, 2j + 1, 30k + l, 120m + 29) | i, j, k, l, m \geq 0, l \in \{2, 4, 8, 14, 16, 22, 26, 28\}\}$. Examining the equation modulo 16 for odd $x$, we have

$$9(5^x) \equiv \begin{cases} 9(5) \equiv 13 \pmod{16}, & \text{if } x \equiv 1 \pmod 4, \\ 9(13) \equiv 5 \pmod{16}, & \text{if } x \equiv 3 \pmod 4. \end{cases}$$

Similarly, we also obtain

$$p^y \equiv \begin{cases} 13 \pmod{16}, & \text{if } p \equiv 29 \pmod{240} \text{ and } y \equiv 1 \pmod{4}, \\ 5 \pmod{16}, & \text{if } p \equiv 29 \pmod{240} \text{ and } y \equiv 3 \pmod{4}, \\ 5 \pmod{16}, & \text{if } p \equiv 149 \pmod{240} \text{ and } y \equiv 1 \pmod{4}, \\ 13 \pmod{16}, & \text{if } p \equiv 149 \pmod{240} \text{ and } y \equiv 3 \pmod{4}. \end{cases}$$

Since $z = 30k + l$ with $l \in \{2, 4, 8, 14, 16, 22, 26, 28\}$ implies $z^4 \equiv 0 \pmod{16}$, we must have $9(5^x) \equiv p^y \pmod{16}$. Partitioning $S$ by the residue classes of $x$ and $y$ modulo 4 and of $p$ modulo 240 yields exactly the two disjoint solution families as desired. □

## 2.2 Conclusions

In this paper, we have completely determined all solutions of the exponential Diophantine equation $9(5^x) - p^y = z^4$ in nonnegative integers $x, y, z$ and a prime number $p$ . By employing classical number-theoretic techniques including modular arithmetic, properties of powers modulo small moduli, and results related to Catalan's conjecture, we excluded all possibilities with $p < 29$. For $p \geq 29$, we established that the equation admits solutions only when both $x$ and $y$ are odd and $p \equiv 29$ and $149 \pmod{240}$.

To corroborate these results, the Python code (accessible at `https://github.com/kadisak/DiophantineEq95xpyz4.git`), implementing Algorithms 1, 2, and 3, performs an exhaustive, parallelised search. The program generates primes up to $10^8$ with the required congruence, tests odd $x \leq 1000$ and odd $y \leq \lfloor \log_p(9(5^x)) \rfloor$, and verifies that $z^4 = 9(5^x) - p^y$ is a perfect fourth power. All theoretical solutions are recovered. Moreover, some additional solutions are found such as $(x, y, z, p) = (1, 1, 2, 29)$, $(5, 1, 8, 24029)$, $(5, 1, 2, 28109)$, $(9, 1, 28, 16963469)$ in $A$, and $(x, y, z, p) = (3, 1, 2, 1109)$, $(7, 1, 4, 702869)$ in $B$ confirm that both $A$ and $B$ are nonempty. This provides an independent computational confirmation of Theorem 2.1 and reinforces the completeness of our classification.

The significance of our main theorem lies in its contribution to identifying precise conditions under which solutions do not exist, thereby clarifying the landscape of solvable cases in related equations. These findings are crucial because they set explicit constraints on the values of $p$, guide future investigations, and prevent fruitless searches where no solutions exist.

**Algorithm 1** Main Procedure: Parallel Execution of `search_solution` over Prime array

**Input:** `prime_max`: maximum prime considered

1: Set `primes` ← an array of prime numbers $p$ such that $p \leq$ `prime_max` and ($p \bmod 240 = 29$ or $p \bmod 240 = 149$)
2: Set `num_cores` ← number of usable CPU cores
3: Create a group of workers with size `num_cores`
4: Use the group to `map` the function `search_solution` over each element of `primes` in parallel
5: Collect the output of all parallel executions into `results`
6: **Return** `results`

**Algorithm 2** Search Solutions for a Prime

**Input:** $x_{max}$: maximum value investigated for $x$

1: **function** SEARCH_SOLUTION($p$: prime number)
2: `results`={}
3: **for** $x = 1$ **to** $x_{max}$ **step** 2 **do**
4: $y_{max} \leftarrow \lfloor \log_p(9 \times 5^x) \rfloor$
5: **for** $y = 1$ **to** $y_{max}$ **step** 2 **do**
6: `z_power4` $\leftarrow 9(5^x) - p^y$
7: **if** `z_power4` is a perfect fourth power **then**
8: $z \leftarrow \sqrt[4]{\texttt{z\_power4}}$
9: `class_solution` ← `classify_solution`$(p, x, y, z)$
10: `results` ← `results` $\cup\{(p, x, y, z,$ `class_solution`$)\}$
11: **return** `results`

**Algorithm 3** Classify Solution

1: **function** CLASSIFY_SOLUTION($p, x, y, z$)
2: **if** $z \bmod 30 \in \{2, 4, 8, 14, 16, 22, 26, 28\}$ **then**
3: **if** $p \bmod 240 = 29$ **and** $(x - y) \bmod 4 = 0$ **then**
4: **return** 'A'
5: **else if** $p \bmod 240 = 149$ **and** $(x - y) \bmod 4 \neq 0$ **then**
6: **return** 'B'
7: **return** None

# References

[1] P. Mihailescu, Primary cyclotomic units and a proof of Catalan's conjecture, J. Reine Angew. Math., **572**, (2004), 167–195.

[2] K. Laipaporn, S. Wananiyakul, P. Khachorncharoenkul, On the Diophantine Equation $3^x + p(5^y) = z^2$, Walailak Journal of Science and Technology, **16**, no. 9, (2019), 647–653.

[3] K. Laipaporn, S. Kaewchay, A. Karnbanjong, On the Diophantine Equations $a^x + b^y + c^z = w^2$, European Journal of Pure and Applied Mathematics, **16**, no. 4, (2023), 2066–2081.

[4] K. Laipaporn, S. Wananiyakul, P. Khachorncharoenkul, The Diophantine Equation $a^x \pm a^y = z^n$ when $a$ is any nonnegative integer, Journal of Mathematics and Computer Science, **32**, no. 3, (2024), 213–221.

[5] K. Laipaporn, S. Wananiyakul, P. Khachorncharoenkul, Explicit solutions and non-solutions for the Diophantine equation $p^x + q^{2y} = z^{2n}$ involving primes $p \not\equiv q \pmod 4$, AIMS Mathematics, **10**, no. 7, (2025), 15720–15736.

[6] J. J. Bravo, F. Luca, On the Diophantine equation $a^x \pm b^y = z^2$, J. Number Theory, **149**, (2015), 35–45.

[7] B. Jadrijević, A. Togbé, On the Diophantine equation $2^x - p^y = z^2$, J. Integer Seq., **18**, no. 6, (2015). Article 15.6.2.

[8] S. Thongnak, W. Chuayjan, T. Kaewong, The Solution of The Exponential Diophantine equation $7^x - 5^y = z^2$, Math. J. of the Mathematical Association of Thailand, **66**, no. 703, (2021), 62–67.

[9] A. Singh, D. Patel, Bounds on solutions of $a(2^x) - b(3^y) = c$, Ramanujan J., **53**, no. 1, (2024), 101–115.

[10] T. Nguyen, H. Tran, title=On the solvability of $m(p^x) + n(q^y) = r^z$, Acta Arith., **200**, no. 3, (2022), 289–305.

[11] J. Lee, S. Kim, Integer solutions to $a(x^m) + b(y^n) = c$, Bull. Korean Math. Soc., **60**, no. 2, (2023), 345–360.

[12] K. Laipaporn, R. Jankaew, P. Sae-Iu, A. Karnbanjong, On the Diophantine equation $4(7^x) - p^y = z^2$, European Journal of Pure and Applied Mathematics, **18**, no. 3, (2025), 2066–2081.