International Journal of Mathematics and Computer Science
Volume **20,** Issue no. 4, (2025), 1079–1102
DOI: https://doi.org/10.69793/ijmcs/04.2025/estrada

Some Results on the Edge Cover Coloring Number Problem of Graphs

Tjaart Jan B. Estrada

Don Mariano Marcos Memorial State University
La Union, Philippines

email: tjestrada@dmmmsu.edu.ph

(Received September 1 2025, Accepted October 24, 2025,
Published October 28, 2025)

Abstract

Let G be a finite undirected simple graph with chromatic index $\chi'(G)$. For a minimal proper edge coloring $\psi : E(G) \to \{1, 2, ..., \chi'(G)\}$ in G, an edge covering set S is an edge covering coloring set if each color class in $\{1, 2, ..., \chi'(G)\}$ appears in some edge in S. The edge cover coloring number of G, denoted by $\chi'_\alpha(G)$, is the minimum cardinality taken over all possible edge cover coloring sets and all minimal proper edge colorings in G. This paper discusses the problem of finding the edge cover coloring number in graphs, particularly for paths, cycles, complete graphs, wheels, and (n, k)-tadpole graphs.

1 Introduction

Edge colorings are of equal importance as vertex colorings. Among the popular applications of edge colorings are scheduling of events [1] and generating course time tables [2]. The concept of edge coverings also catches much attention in graph theory, since it provides a number of applications. The problem of finding a minimum vertex or edge covering set, has practical applications in optimization problems. Finding the minimum number of police officers to be placed at various street intersections, so that every street is watched,

Key words and phrases: Edge Coloring, Edge Covering, Edge Cover Coloring Set, Edge Cover Coloring Number.
AMS (MOS) Subject Classifications: 05C15, 05C38, 05C70.
ISSN 1814-0432, 2025, https://future-in-tech.net

is in fact a simple application of graph covering. Graph covering also has its uses in the computation biochemistry, particularly in single nucleotide polymorphism (or "SNP") assembly problems. The relationships between the given SNPs are summarized by a graph, and conflict between the SNPs are removed by identifying a minimal vertex covering of the graph [3].

The work of Chaluvaraju, Nandeesh Kumar, and Rusagara [4] combines the theories of coloring and covering in graphs through the concept of a *vertex cover coloring set*, and the *vertex cover chromatic number*. A vertex cover coloring set of G is a vertex cover C of G such that there exists a proper vertex coloring of G using $\chi(G)$ colors, and each color class appears in some vertex of C. The cardinality of the smallest such vertex cover coloring set, is referred to as the vertex cover chromatic number, denoted by $\chi_\alpha(G)$. Exact values of $\chi_\alpha(G)$ for some standard graphs were obtained, and relationships between $\chi_\alpha(G)$ to some graph theoretic parameters were established in [4].

In this paper, we present results on the edge-counterpart of the vertex cover chromatic number $\chi_\alpha(G)$ introduced in [4], defined by the parameter $\chi'_\alpha(G)$.

2 The Edge Cover Coloring Number

Let G be a finite undirected graph with chromatic index $\chi'(G)$. An *edge cover coloring set* S of G, is an edge covering of G such that for some proper edge coloring $\psi : E(G) \to \{1, 2, ..., \chi'(G)\}$, each color in $\{1, 2, ..., \chi'(G)\}$ appears in some edge of S. This definition is illustrated using a minimal proper edge-coloring of the Petersen graph P shown below.

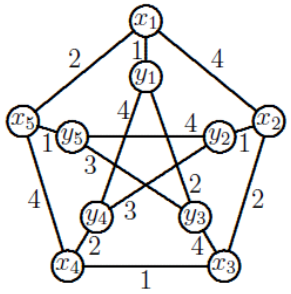

Figure 1: A 4-edge coloring of the Petersen Graph

The set $S_1 = \{x_1y_1, x_2y_2, x_3y_3, x_4y_4, x_5y_5\}$ is an edge covering of P, how-

ever it is not an edge cover coloring set since the color class 3 does not appear in some edge of S_1. The set $S_2 = \{x_1x_2, x_3x_4, x_5y_5, y_1y_3, y_2y_4\}$ is an edge cover coloring set. Five edges are required to obtain an edge covering P, and so S_2 is an edge cover coloring set with minimum cardinality. We refer to the minimum cardinality of an edge cover coloring set among all proper edge-colorings of G, as the *edge cover coloring number*, denoted by the parameter $\chi'_\alpha(G)$. Thus for the Petersen graph, we have $\chi'_\alpha(P) = 5$.

It is important to note that $\rho(G) \leq \chi'_\alpha(G)$ since an edge cover coloring set must be an edge covering. Also, since each of the colors in $\{1, 2, ..., \chi'(G)\}$ must appear in some edge, it can be deduced that $\chi'(G) \leq \chi'_\alpha(G)$.

3 Edge Cover Coloring Numbers of Some Named Graphs

We discuss the edge cover coloring number of some named graphs, in particular, paths, cycles, complete graphs, wheels, and tadpole graphs.

Theorem 3.1. *If G is a path P_n, then $\chi'_\alpha(G) = \lceil \frac{n+1}{2} \rceil$.*

Proof. Suppose that the vertex set of P_n is $\{v_1, v_2, ..., v_n\}$, and its edges are given by $e_i = v_iv_{i+1}$, where $1 \leq i \leq n-1$. If $\psi(G) : E(G) \to \{1, 2\}$ is a proper edge coloring of P_n, then the edges are colored alternately in 1 and 2. Without loss of generality, let $\psi(e_1) = 1$. If n is even, then $\psi(e_1) = \psi(e_{n-1}) = 1$. Thus the larger edge color class is colored in 1. Note that the set $\{e_i | \, i \text{ is odd}\}$ is a minimum edge cover of P_n, however one edge is required from the edge color 2. Thus consider the set $S = \{e_i | \, i \text{ is odd}\} \cup \{e_j\}$, where j is even. S is a minimal edge cover coloring set, and so $\chi'_\alpha(G) = |S| = \frac{n}{2} + 1$. If n is odd, then e_1 and e_{n-1} are colored in 1 and 2, respectively. In this case, the two edge color classes are of equal cardinality. A minimal edge cover coloring set is $S = \{e_i | \, i \text{ is odd}\} \cup \{e_{n-1}\}$. Therefore we have $\chi'_\alpha(G) = |S| = \frac{n-1}{2} + 1$. The results for the two cases simplify to $\chi'_\alpha(G) = \lceil \frac{n+1}{2} \rceil$. □

Theorem 3.2. *If G is a cycle C_n, then* $\chi'_\alpha(G) = \begin{cases} 3 & \text{if } n=3 \\ \lceil \frac{n+1}{2} \rceil & \text{if } n \geq 4 \end{cases}$

Proof. Let $\{e_1, e_2, ..., e_n\}$ be the edge set of C_n, where two consecutive edges, and e_1, e_n, are incident. If n is even, then $\chi'(G) = 2$. Color the edges of C_n with $\psi(e_t) = 1$ if t is odd and $\psi(e_t) = 2$ otherwise. Each color class has $\frac{n}{2}$

edges. Pick one edge color class, say those colored in 1, and pick one edge from the edges colored in 2, say e_2. That is, let $S = \{e_i | \, i \text{ is odd}\} \cup \{e_2\}$. This is a minimal edge cover coloring set with cardinality $\chi'_\alpha(G) = \frac{n}{2} + 1 = \frac{n+2}{2}$. If n is odd, then $\chi'(G) = 3$. For $n = 3$, the edge set of C_3 is the only possible edge cover coloring set and so $\chi'_\alpha(G) = 3$. For $n \neq 3$, define the proper edge-coloring $\psi : E(G) \to \{1, 2, 3\}$ as

$$\psi(e_i) = \begin{cases} 1 & \text{if } i = 3k+1, i \neq n \\ 3 & \text{if } i = 3k \\ 2 & \text{otherwise} \end{cases}$$

Consider the set $S = \{e_i | \, i \text{ is odd}\}$. Note that each color class is represented in S. Moreover, $|S| = \frac{n+1}{2}$, which is also the minimum number of edges required in any edge covering of G of odd order. Thus S is a minimal edge cover coloring set, and $\chi'_\alpha(G) = \frac{n+1}{2}$. The results in these two cases now give the desired formula for $\chi'_\alpha(G)$. □

Theorem 3.3. *If G is a complete graph K_n, $n \geq 4$, then* $\chi'_\alpha(G) = \begin{cases} n-1 \text{ if } n \text{ even} \\ n \text{ if } n \text{ odd} \end{cases}$

Proof. It is well known that for any complete graph, $\chi'(G) = n - 1$ if n is even, and $\chi'(G) = n$ otherwise. Let $\{v, v_1, v_2, ..., v_{n-1}\}$ be the vertex set of K_n. If n is even, consider the edge covering $S = \{vv_1 | \, 1 \leq i \leq n-1\}$ for some minimal proper edge-coloring ψ of K_n. Since each edge $e \in S$ is incident with the same vertex v, there exists a unique $w \in \{1, 2, ..., n-1\}$, such that $\psi(e) = w$ for each e. Thus S is a minimal edge cover coloring set since $|S| = \chi'(K_n)$, and so the result $\chi'_\alpha(G) = n - 1$.

Now suppose n is odd. Every edge of the form $e = vv_i, 1 \leq i \leq n-1$, is incident with the same vertex v, thus there is a unique $w \in \{1, 2, ..., n\}$ such that $\psi(e) = w$ for each e. Moreover, this set of edges form an edge covering of K_n. However there is exactly one color class $x \in \{1, 2, ..., n\}$ which does not appear in any of the edges vv_i. Thus consider $S = \{vv_i | \, 1 \leq i \leq n-1\} \cup \{y\}$, with $\psi(y) = x$. Now S is a minimal edge cover coloring set with cardinality $|S| = (n-1) + 1 = n = \chi'(K_n)$. Thus $\chi'_\alpha(G) = n$. □

Theorem 3.4. *If G is a wheel graph of order $n + 1$, then $\chi'_\alpha(G) = n$.*

Proof. Let v be the central vertex of the wheel graph W_n, and suppose the set of edges $\{vv_i | 1 \leq i \leq n\}$ are the spokes of W_n. Note that this set of edges is

an edge covering of W_n. If $\psi : E(G) \to \{1, 2, ..., n\}$ is a minimal proper edge coloring of W_n, then the set $\{vv_i | 1 \leq i \leq n\}$ is a minimal edge cover coloring set since each of the n colors appear in this set. Thus $\chi'_\alpha(G) = |S| = n$. □

In an (n, k)-tadpole graph, deleting the vertex of degree 3 and its incident edges forms two paths P_{n-1} and P_k. Thus Theorem 3.1 may be utilized to obtain the general formula for $\chi'_\alpha(G)$ of this graph.

Corollary 3.5. *If G is the (n, k)-tadpole graph, $n \geq 4$, then* $\chi'_\alpha(G) = \lceil \frac{n}{2} \rceil + \lceil \frac{k}{2} \rceil + 1$.

Proof. Let v be the vertex of degree 3. Assign $v_1, v_2, ..., v_k$ and $v, u_1, u_2, ..., u_{n-1}$ to be the vertex sequence for the path P_k and cycle C_n, respectively. Color the edge vv_1 with 1, and the edges of the path P_k as follows: $\psi(v_i v_{i+1}) = 2$ if i is even, and $\psi(v_i v_{i+1}) = 3$ otherwise, where $1 \leq i \leq k$. Also if n is even, the edges of the cycle C_n is colored with $\psi(u_i u_{i+1}) = 2$ if i is even and $\psi(u_i u_{i+1}) = 3$ otherwise, where $1 \leq i \leq n - 2$. If C_n has odd order n, then $\psi(u_{n-2} u_{n-1}) = 1$, $\psi(u_{n-1} v) = 3$ and the rest of the edges are colored as 2 or 3 in the same manner as the previous case. The edge coloring ψ is minimal since the maximum degree is 3.

From Theorem 3.1, a minimal edge cover coloring set of P_{k-1} must have $\lceil \frac{k}{2} \rceil$ vertices. By deleting the vertex v together with the incident edges, the path P_{n-1} is formed. By Theorem 3.1, a minimal edge cover coloring set for P_{n-1} requires $\lceil \frac{n}{2} \rceil$ vertices. Thus to construct a minimal edge cover coloring set for the (n, k)-tadpole graph, choose all edges from the minimal edge cover coloring sets from P_{k-1}, P_{n-1}, and the edge vv_1. This implies the desired result: $\chi'_\alpha(G) = \lceil \frac{n}{2} \rceil + \lceil \frac{k}{2} \rceil + 1$. □

4 Conclusion

The edge cover coloring number problem was tackled for paths, cycles, complete graphs, wheels and (n, k)-tadpole graphs. The problem can be explored further for larger and more complex graphs, as well as graphs resulting from standard graph theoretic operations.

References

[1] S. Skiena, The Algorithm Design Manual, Springer-Verlag, London, 2008.

[2] V. Ramaswamy, B. Poornima, Application of Edge Coloring of a Fuzzy Graph, International Journal of Computer Applications, **6**, no. 2, (2010).

[3] G. Lancia, V. Afna, S. Istrail, L. Lippert, R. Schwartz, SNPs Problems, Complexity and Algorithms, Springer-Verlag, (2001), 182–193.

[4] B. Chaluvaraju, C. Nandeesh Kumar, I. Rusagara, The Vertex Cover Chromatic Number of a Graph, Canadian Journal on Computing Mathematics, Natural Sciences, Engineering and Medicine, **2**, no. 1, (2011).