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On the Biquadratic Diophantine Equation $x^4 + 9x^2y^2 + 27y^4 = z^2$

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Abstract

We demonstrate that the biquadratic Diophantine equation $x^4 + 9x^2y^2 + 27y^4 = z^2$ admits no non-trivial positive integer solutions. Employing a Fermat-style infinite descent, our proof combines congruences modulo 8 and 9, 3-adic valuations, and three distinct difference-of-squares factorizations to reveal local obstructions, culminating in the descent argument. This approach not only solves the equation but also exemplifies how tailored algebraic identities can unlock solutions to challenging quartic Thue equations.

1 Introduction

Quartic forms of the shape $x^4+ax^2y^2+by^4 = z^2$ are classical; local arguments and descent often suffice (see [1], [2]) and, when needed, elliptic methods apply in broader families (e.g. [3, 4, 5]). We solve

$$x^4 + 9x^2y^2 + 27y^4 = z^2 \tag{1.1}$$

by an elementary 3-adic descent optimized for $(a, b) = (9, 27)$.

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Theorem 1.1. *The only integer solutions of* (1.1) *are* $(x, y, z) = (m, 0, \pm m^2)$ *with* $m \in \mathbb{Z}$.

2 Two identities and basic tools

Write $\mathrm{v}_3(\cdot)$ for the 3-adic valuation. Set

$$A := x^2 + 3xy + 9y^2, \qquad B := x^2 - 3xy + 9y^2.$$

A direct expansion yields

$$(z - A)(z + A) = -6y\,U, \quad U := x^3 + 3x^2y + 9xy^2 + 9y^3, \tag{2.2}$$
$$(z - B)(z + B) = \ 6y\,V, \quad V := x^3 - 3x^2y + 9xy^2 - 9y^3. \tag{2.3}$$

Lemma 2.1 (Parity). *If* (1.1) *holds, then* y *is even.*

Proof. Mod 8, fourth powers are 0 or 1. If y is odd, then $x^4 + 9x^2y^2 + 27y^4 \equiv x^4 + x^2 + 3 \in \{3, 5\}$, not a square. □

Lemma 2.2 (Basic 3-adic data). *Assume* (1.1). *With*

$A := x^2 + 3xy + 9y^2$, $B := x^2 - 3xy + 9y^2$, $U := x^3 + 3x^2y + 9xy^2 + 9y^3$,

$V := x^3 - 3x^2y + 9xy^2 - 9y^3$, *we have:*

1. *If* $3 \mid x$ *and* $3 \nmid y$, *then* $\mathrm{v}_3(A) = \mathrm{v}_3(B) = 2$, $\mathrm{v}_3(U) = \mathrm{v}_3(V) = 2$, $\mathrm{v}_3(z) = 1$.
2. *If* $3 \mid y$ *and* $3 \nmid x$, *then* $\mathrm{v}_3(A) = \mathrm{v}_3(B) = \mathrm{v}_3(U) = \mathrm{v}_3(V) = \mathrm{v}_3(z) = 0$.
3. *If* $3 \mid x$ *and* $3 \mid y$, *then* $9 \mid z$.

Proof. Write v_3 for the 3-adic valuation.
(1) $3 \mid x$**,** $3 \nmid y$**.** Put $x = 3x_1$. Then

$$z^2 = 81x_1^4 + 81x_1^2y^2 + 27y^4 = 27\big(3x_1^4 + 3x_1^2y^2 + y^4\big),$$

whose bracket $\equiv y^4 \not\equiv 0 \pmod 3$; hence $\mathrm{v}_3(z) = 1$. Next,

$$U = 27(\cdots) + 9y^3 \equiv 9y^3 \not\equiv 0 \pmod{27}, \qquad V = 27(\cdots) - 9y^3 \equiv -9y^3 \not\equiv 0 \pmod{27},$$

so $\mathrm{v}_3(U) = \mathrm{v}_3(V) = 2$. Also

$$A = 9(x_1^2 + x_1y + y^2), \quad B = 9(x_1^2 - x_1y + y^2) \Rightarrow \mathrm{v}_3(A), \mathrm{v}_3(B) \geq 2.$$

If $\mathrm{v}_3(A) \geq 3$ then, using $(z - A)(z + A) = -6yU$ and $3 \nmid y$,

$$\mathrm{v}_3\big((z - A)(z + A)\big) = \mathrm{v}_3(6y) + \mathrm{v}_3(U) = 3,$$

but with $\mathrm{v}_3(z) = 1$ and $\mathrm{v}_3(A) \geq 3$ both $z \pm A$ have valuation 1, giving 2 which is a contradiction. Hence $\mathrm{v}_3(A) = 2$, and similarly $\mathrm{v}_3(B) = 2$.

(2) $3 \mid y$**,** $3 \nmid x$**.** Put $y = 3y_1$. Then

$$A \equiv B \equiv x^2 \not\equiv 0 \pmod 3, \quad U \equiv V \equiv x^3 \not\equiv 0 \pmod 3, \quad z^2 \equiv x^4 \not\equiv 0 \pmod 3,$$

so all five valuations are 0.

(3) $3 \mid x, y$**.** With $x = 3X$, $y = 3Y$,

$$z^2 = 81\big(X^4 + X^2Y^2 + 27Y^4\big) \Rightarrow \mathrm{v}_3(z) \geq 2;$$

i.e., $9 \mid z$. □

3 The 3-link $3 \mid x \iff 3 \mid y$

Lemma 3.1. *If* (1.1) *holds, then* $3 \mid x$ *if and only if* $3 \mid y$.

Proof. ($\Rightarrow$) If $3 \mid x$ and $3 \nmid y$, Lemma 2.2(1) gives $\mathrm{v}_3(A) = \mathrm{v}_3(B) = 2$ and $\mathrm{v}_3(z) = 1$. Thus $z \pm A$ and $z \pm B$ are all 3-multiples but none is 9-multiple, contradicting (2.2)–(2.3) where $\mathrm{v}_3((z \mp A)(z \pm A)) = \mathrm{v}_3(6y) + \mathrm{v}_3(U) = 3$ (and similarly for B).

($\Leftarrow$) If $3 \mid y$ and $3 \nmid x$, then modulo 27 we have $z^2 \equiv x^4$, hence $z \equiv \pm x^2 \pmod{27}$. Also $A \equiv B \equiv x^2 \pmod{27}$, so exactly one of $z \pm A$ (and one of $z \pm B$) is divisible by 27. From (2.2)–(2.3) and Lemma 2.2 (2),

$$\mathrm{v}_3\big((z - A)(z + A)\big) = \mathrm{v}_3(6y) + \mathrm{v}_3(U) = 1 + \mathrm{v}_3(y),$$

yet the preceding congruence forces $\mathrm{v}_3\big((z - A)(z + A)\big) \geq 3$, so $\mathrm{v}_3(y) \geq 2$. Replacing y by $y/9$ in the same argument (keeping x fixed mod 3) repeats the contradiction; hence $3 \mid x$. □

4 Normalization, descent, and proof of Theorem 1.1

By Lemma 3.1, any nontrivial solution has $3 \mid x, y$. Write

$$x = 3X, \qquad y = 3Y, \qquad z = 9Z.$$

Dividing (1.1) by 81 gives the scaled equation

$$Z^2 = X^4 + X^2Y^2 + 27Y^4. \tag{4.4}$$

It enjoys identities paralleling (2.2)–(2.3):

$$(Z - A_1)(Z + A_1) = -2Y\,U_1, \quad (Z - B_1)(Z + B_1) = 2Y\,V_1, \tag{4.5}$$

with $A_1 := X^2 + XY + 9Y^2$, $B_1 := X^2 - XY + 9Y^2$, and $U_1 := X^3 + 9X^2Y + 9XY^2 + 27Y^3$, $V_1 := X^3 - 9X^2Y + 9XY^2 - 27Y^3$.

Lemma 4.1 (Link for the scaled equation). *If* (4.4) *holds, then* $3 \mid X$ *iff* $3 \mid Y$.

Proof. As in Lemma 3.1, using (4.5) in place of (2.2)–(2.3): when $3 \mid Y$, $3 \nmid X$, we have $Z \equiv \pm X^2$ and $A_1 \equiv B_1 \equiv X^2 \pmod{27}$, forcing $\mathrm{v}_3((Z \mp A_1)(Z \pm A_1)) \geq 3$ while $\mathrm{v}_3((Z \mp A_1)(Z \pm A_1)) = \mathrm{v}_3(2Y) = 1$. The opposite case is similar. □

Proof of Theorem 1.1. Assume a nontrivial solution with $|y|$ minimal. By Lemmas 3.1 and 2.2 (3), $x = 3X$, $y = 3Y$, $z = 9Z$ with (X, Y, Z) satisfying (4.4). By Lemma 4.1, $3 \mid X, Y$, so the same normalization applied to (4.4) produces another integer solution with Y replaced by $Y/3$, strictly decreasing $\mathrm{v}_3(y)$ and contradicting minimality. Hence $y = 0$, and $z = \pm x^2$. □

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References

[1] L. E. Dickson, *History of the Theory of Numbers, Vol. II: Diophantine Analysis*, Carnegie Institution of Washington, 1920.

[2] L. J. Mordell, *Diophantine Equations*, Academic Press, 1969.

[3] P. G. Walsh, On equations of the form $x^4 + ax^2y^2 + by^4 = z^2$, *Acta Arith.,* **63,** (1993), 145–156.

[4] K. Zelator, A family $x^4 + 2nx^2y^2 + my^4 = z^2$ with no solutions in $(\mathbb{Z}^+)^3$, *arXiv:0905.3346,* (2009).

[5] M. Ulas, On $x^4 + y^4 + kx^2y^2 = z^2$, *Int. Math. Res. Not.,* (2021), 1–23.