International Journal of Mathematics and Computer Science
Volume **20,** Issue no. 4, (2025), 1103–1107
DOI: https://doi.org/10.69793/ijmcs/04.2025/nilsrakoo

A Reconsideration on Infinite Solutions of the Diophantine Equation $a^x + (2a)^y = z^2$ where $a \equiv 3 \pmod{16}$

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(Received October 3, 2025, Accepted October 30, 2025,
Published October 31, 2025)

Abstract

In this paper, we revisit the infinite solutions of the Diophantine equation $a^x + (2a)^y = z^2$ where $a \equiv 3 \pmod{16}$, originally studied by Rangpung. His work given that the equation has the three forms of non-negative integer solutions. Our finding exposes additional solutions and gives a better version of his result.

1 Introduction

In [2], Rangpung showed that all solutions of the Diophantine equation

$$a^x + (2a)^y = z^2 \tag{1.1}$$

where $a \equiv 3 \pmod{16}$ and display them as:

Key words and phrases: Diophantine equation, infinite solutions.
AMS (MOS) Subject Classifications: 11D61, 11B39.
The corresponding author is Achariya Nilsrakoo.
ISSN 1814-0432, 2025, https://future-in-tech.net

(i) $(x, y, z) = (1, 0, \sqrt{a+1})$ with $a = 64t^2 \pm 32t + 3$,

(ii) $(x, y, z) = (1, 1, \sqrt{3a})$ with $a = 192t^2 \pm 48t + 3$, and

(iii) $(x, y, z) = (2, 3, \sqrt{8a^3 + a^2})$ with $a = 512t^2 \pm 80t + 3$,

where x, y, z are non-negative integers and t is an integer. Unfortunately, the proof contains an oversight. More precisely, reasoning in Case 9.2 and Case 9.4 is *not enough* to guarantee that the equation has no non-negative integer solutions. In this paper, we reveal additional solutions and suggest a better version of [2, Theorem 3.1]. Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{N}_0$ the sets of positive integers and non-negative integers, respectively. We need the following results.

Lemma 1.1. *Let $a, x, y \in \mathbb{N}_0$ be such that $a \equiv 3 \pmod{16}$, x is odd and y is even. Then*

(i) $x^2 \equiv 1 \pmod 4$, $y^2 \equiv 0 \pmod 4$, $a^x \equiv 3 \pmod 4$ *and* $a^y \equiv 1 \pmod 4$.

(ii) $x^2 \equiv 1, 9 \pmod{16}$, $y^2 \equiv 0, 4 \pmod{16}$, $a^x \equiv 3, 11 \pmod{16}$ *and* $a^y \equiv 1, 9 \pmod{16}$.

Lemma 1.2 ([1, Catalan's conjecture, Mihăilescu's theorem]). *Let $a, b, x, y \in \mathbb{N}$ be such that $\min\{a, b, x, y\} \geq 2$. The equation $a^x - b^y = 1$ has the unique solution $(a, b, x, y) = (3, 2, 2, 3)$.*

Lemma 1.3 (Pythagorean theorem). *Let $x, y, z \in \mathbb{N}$. Then (x, y, z) with $\gcd(x, y, z) = 1$ and $x^2 + y^2 = z^2$ if and only if there are $u, v \in \mathbb{N}$ with $u > v$, $\gcd(u, v) = 1$ and $u \not\equiv v \pmod 2$ such that $x = u^2 - v^2$, $y = 2uv$ and $z = u^2 + v^2$.*

2 Main results

A better version of [2, Theorem 3.1] follows.

Theorem 2.1. *The Diophantine equation* (1.1) *has the following non-negative integer solutions (x, y, z):*

(i) $(x, y, z) = (6, 4, 45)$ *with* $a = 3$.

(ii) $(x, y, z) = (1, 0, 4t + 2)$ *with* $a = 16t^2 + 16t + 3$, $t \in \mathbb{N}_0$.

(iii) $(x, y, z) = (1, 1, b)$ *with*

- $a = 192t^2 + 48t + 3$ *and* $b = 24t + 3$, $t \in \mathbb{N}_0$;
- $a = 192t^2 + 336t + 147$ *and* $b = 24t + 21$, $t \in \mathbb{N}_0$.

(iv) $(x, y, z) = (2\ell, 2\ell + \kappa, a^\ell b)$ *with*

- $a^\kappa = 2^{2\ell+\kappa-2}q^2 + q$ *and* $b = 2^{2\ell+\kappa-1}q + 1$;
- $a^\kappa = 2^{2\ell+\kappa-2}q^2 + (2^{2\ell+\kappa-1} - 1)q + 2^{2\ell+\kappa-2} - 1$ *and* $b = 2^{2\ell+\kappa-1}q + 2^{2\ell+\kappa-1} - 1$.

where $\ell, \kappa, q \in \mathbb{N}_0$ *with* $\ell > 0$ *and* κ *is odd. In particular, if* $\kappa = 1$ *and* $t \in \mathbb{N}_0$, *then*

(a) $(x, y, z) = (2, 3, ab)$ *with*

- $a = 512t^2 + 80t + 3$ *and* $b = 64t + 5$;
- $a = 512t^2 + 944t + 435$ *and* $b = 64t + 59$;

(b) $(x, y, z) = (4, 5, a^2b)$ *with*

- $a = 2048t^2 + 1264t + 195$ *and* $b = 256t + 79$;
- $a = 2048t^2 + 2832t + 979$ *and* $b = 256t + 177$;

(c) $(x, y, z) = (2\ell, 2\ell + 1, a^\ell b)$, *where* $\ell \geq 3$ *with*

- $a = 2^{2\ell-1}(16t + 3)^2 + 16t + 3$ *and* $b = 2^{2\ell}(16t + 3) + 1$;
- $a = 2^{2\ell-1}(16t + 15)^2 + (2^{2\ell} - 1)(16t + 15) + 2^{2\ell-1} - 1$ *and* $b = 2^{2\ell}(16t + 15) + 2^{2\ell} - 1$.

Proof. This result is divided into three cases for x and y.

Case 1. Consider $x > y$. Then $a^y\,(a^{x-y} + 2^y) = z^2$. We know that $\gcd(a^y, a^{x-y} + 2^y) = 1$. There are odd $z_1, z_2 \in \mathbb{N}$ such that $z = z_1 z_2$, $a^y = z_1^2$ and $a^{x-y} + 2^y = z_2^2$. By Lemma 1.1, we have $a^y = z_1^2 \equiv 1, 9 \pmod{16}$. Therefore, y is even. Assume that $y = 2i$ where $i \in \mathbb{N}_0$. Then

$$a^{x-2i} + 2^{2i} = z_2^2. \tag{2.2}$$

Case 1.1. Assume that x is even. Then $x = 2j$ where $j \in \mathbb{N}_0$ and

$$(a^{j-i})^2 + (2^i)^2 = z_2^2.$$

Notice that $\gcd(a^{j-i}, 2^i) = 1$. By Lemma 1.3, there are $u, v \in \mathbb{N}$ with $u > v$, $\gcd(u, v) = 1$ and $u \not\equiv v \pmod 2$ such that $a^{j-i} = u^2 - v^2$, $2^i = 2uv$ and $z_2 = u^2 + v^2$. It is known that $i > 1$, $u = 2^{i-1}$, $v = 1$, $a^{j-i} = 2^{2(i-1)} - 1$ and $z_2 = 2^{2(i-1)} + 1$. If $i \geq 3$, then $2^{2(i-1)} \equiv 0 \pmod{16}$ and so $a^{j-i} \equiv 15$

(mod 16). Which contradicts with Lemma 1.1. This implies that $i = 2$. Then $a^{j-2} = 3$ and $z_2 = 5$. So, we get $a = 3$ and $j = 3$. This means $x = 6, y = 4$ and $z = 3^2 \cdot 5 = 45$. Then $(x, y, z) = (6, 4, 45)$ with $a = 3$.

Case 1.2. Assume that x is odd. Since $a \equiv 3 \pmod{16}$, we have $a \equiv 3 \pmod 4$ and so $a^{x-2i} \equiv 3 \pmod 4$. From (2.2), we have $z_2^2 \equiv 3 + 0 \equiv 3 \pmod 4$ for all $i > 0$, which is impossible. This implies that $y = 2i = 0$. Therefore, $z_2^2 - a^x = 1$ with $a \geq 3$ and $z_2 \geq 2$. By Lemma 1.2, we have $x = 1$. That is, $z^2 = a + 1$ and $z^2 \equiv 4 \pmod{16}$. This implies that $z \equiv 2 \pmod 4$. Setting $z = 4t + 2$ for $t \in \mathbb{N}_0$, we get $a = 16t^2 + 16t + 3$. Then $(x, y, z) = (1, 0, 4t + 2)$ with $a = 16t^2 + 16t + 3$.

Case 2. Consider $x = y$. If $x = y = 2k$ where $k \in \mathbb{N}_0$, then

$$a^{2k}\left(1 + 2^{2k}\right) = z^2. \tag{2.3}$$

Let $\gcd(a^k, z) = d$. Then $a^k = dr$ and $z = ds$, where $r, s \in \mathbb{N}$ with $\gcd(r, s) = 1$. It follows from (2.3) and $\gcd(r^2, s^2) = 1$ that $r = 1$ and $1 + 2^{2k} = s^2$. Then $1 = s^2 - 2^{2k} = (s - 2^k)(s + 2^k)$. This implies that $s = 1$ and $2^k = 0$, which are impossible. Now, we get x is odd. Since $a \equiv 3 \pmod{16}$, we have $a \equiv 3 \pmod 4$ and so $a^x \equiv 3 \pmod 4$. Then $z^2 \equiv 3(1 + 0) \equiv 3 \pmod 4$ for all $x > 1$, which is impossible. This implies that $x = y = 1$. Therefore, $z^2 = 3a$. Then $3|z$ and so $z = 3q$ for $q \in \mathbb{N}$. Moreover, $a = 3q^2$. This implies that $q^2 \equiv 1 \pmod{16}$ and so $q \equiv 1, 7 \pmod 8$. Setting $q = 8t + 1$ or $q = 8t + 7$ where $t \in \mathbb{N}_0$, we get $(x, y, z) = (1, 1, 24t + 3)$ with $a = 192t^2 + 48t + 3$ and $(x, y, z) = (1, 1, 24t + 21)$ with $a = 192t^2 + 336t + 147$.

Case 3. Consider $x < y$. Then

$$a^x\left(1 + 2^y a^{y-x}\right) = z^2. \tag{2.4}$$

Since $a \equiv 3 \pmod{16}$, we have $a \equiv 3 \pmod 4$. If x is odd, then $a^x \equiv 3 \pmod 4$. Since $y > x \geq 1$, we obtain $2^y \equiv 0 \pmod 4$ and so $z^2 \equiv 3 \pmod 4$, which is impossible. Assume that x is even. If $x = 0$, then $z^2 - (2a)^y = 1$ with $2a \geq 6$ and $z \geq 3$. By Lemma 1.2, we have $y = 1$. That is, $z^2 = 2a + 1$. Since $a \equiv 3 \pmod{16}$, we see that $z^2 = 2a + 1 \equiv 7 \pmod{16}$, this is a contradiction. Then $x = 2\ell$ where $\ell \in \mathbb{N}$. We know that $\gcd(a^{2\ell}, 1 + 2^y a^{y-2\ell}) = 1$. From (2.4), there are odd $z_3, z_4 \in \mathbb{N}$ such that $z = z_3 z_4$, $a^{2\ell} = z_3^2$ and

$$1 + 2^y a^{y-2\ell} = z_4^2. \tag{2.5}$$

So we get $z_3 = a^\ell$ and $z = a^\ell z_4$. If y is even, then $y = 2m$ where $m \in \mathbb{N}$ and

$$1 = z_4^2 - (2^m a^{m-\ell})^2 = (z_4 - 2^m a^{m-\ell})(z_4 + 2^m a^{m-\ell}).$$

Therefore, $z_4 = 1$ and $2^m a^{m-\ell} = 0$, which are impossible. This implies that y is odd. Let $y = 2n+1$ and $\kappa = 2(n-\ell)+1$, where $n \in \mathbb{N}$. From (2.5), we have

$$1 + 2^{2n+1}a^\kappa = z_4^2 \quad \text{and} \quad a^\kappa = \frac{z_4^2 - 1}{2^{2n+1}}. \tag{2.6}$$

Then $z_4^2 \equiv 1 \pmod{2^{2n+1}}$ and so $z_4 \equiv \pm 1 \pmod{2^{2n}}$. Setting $b = z_4$, we obtain that there exists $q \in \mathbb{N}_0$ such that

$$(a^\kappa, b) = (2^{2\ell+\kappa-2}q^2 + q, 2^{2\ell+\kappa-1}q + 1),$$
$$(a^\kappa, b) = (2^{2\ell+\kappa-2}q^2 + (2^{2\ell+\kappa-1}-1)q + 2^{2\ell+\kappa-2} - 1, 2^{2\ell+\kappa-1}q + 2^{2\ell+\kappa-1} - 1).$$

In particular, consider $\kappa = 1$. Then

$$(a, b) = (2^{2\ell-1}q^2 + q, 2^{2\ell}q + 1), (2^{2\ell-1}q^2 + (2^{2\ell}-1)q + 2^{2\ell-1} - 1, 2^{2\ell}q + 2^{2\ell} - 1).$$

Case 3.1. Consider $\ell = 1$. Then $x = 2, y = 3$, and

$$(a, b) = (2q^2 + q, 4q + 1), (2q^2 + 3q + 1, 4q + 3).$$

Then $a = 2q^2 + q \equiv 3 \pmod{16}$ if and only if $q \equiv 1 \pmod{16}$, and $a = 2q^2 + 3q + 1 \equiv 3 \pmod{16}$ if and only if $q \equiv 14 \pmod{16}$. Hence, $(a, b) = (512t^2 + 80t + 3, 64t + 5), (512t^2 + 944t + 435, 64t + 59)$, where $t \in \mathbb{N}_0$.

Case 3.2. Consider $\ell = 2$. Then $x = 4, y = 5$, and

$$(a, b) = (8q^2 + q, 16q + 1), (8q^2 + 15q + 7, 16q + 15).$$

Then $a = 8q^2 + q \equiv 3 \pmod{16}$ if and only if $q \equiv 11 \pmod{16}$, and $a = 8q^2 + 15q + 7 \equiv 3 \pmod{16}$ if and only if $q \equiv 4 \pmod{16}$. Hence, $(a, b) = (2048t^2 + 2832t + 979, 256t + 177), (2048t^2 + 1264t + 195, 256t + 79)$, where $t \in \mathbb{N}_0$.

Case 3.3. Consider $\ell \geq 3$. Then $a = 2^{2\ell-1}q^2 + q \equiv 3 \pmod{16}$ if and only if $q \equiv 3 \pmod{16}$ if and only if $q = 16t + 3$ where $t \in \mathbb{N}_0$, and $a = 2^{2\ell-1}q^2 + (2^{2\ell} - 1)q + 2^{2\ell-1} - 1 \equiv 3 \pmod{16}$ if and only if $q \equiv 15 \pmod{16}$ if and only if $q = 16t + 15$ where $t \in \mathbb{N}_0$. □

References

[1] P. Mihăilescu, Primary cyclotomic units and a proof of Catalan's conjecture, J. Reine Angew. Math., **572,** (2004), 167–195.

[2] T. Rangpung, Infinite solutions of the Diophantine equation $a^x + (2a)^y = z^2$ where $a \equiv 3 \pmod{16}$, Int. J. Math. Comput. Sci., **20,** no. 3, (2025), 805–809.