International Journal of Mathematics and Computer Science
Volume **20,** Issue no. 3, (2025), 737–741
DOI: https://doi.org/10.69793/ijmcs/03.2025/omar

e-Prime Ideals of Commutative Rings

Omar A. Abdullah, Ali Sh. Ajeel, Mohmad E. Dahash

Department of Mathematics
College of Computer Science and Mathematics
Tikrit University
Tikrit, Iraq

email: omerabdulrazzaqa@tu.edu.iq, ali.shebl@tu.edu.iq,
mohmad.e.dahash35391@st.tu.edu.iq

(Received April 1, 2025, Accepted April 27, 2025,
Published April 28, 2025)

Abstract

In this paper, we define and study a new class of ideals, called e-prime ideals, that largely generalizes the concept of prime ideals. Moreover, we settle the e-prime ideals of the direct product and polynomial rings.

1 Introduction

In this paper, all rings will be commutative with identity. We let $\mathrm{ID}(R)$ denote the set of all idempotents of R. In recent years, many generalizations of the concept of prime ideal have been studied such as weakly prime ideals [1], S-prime ideals [2], and 1-absorbing prime ideals [3]. In this paper, we study a new class of ideals, called e-prime ideals.
Let R be a ring and let $e \in \mathrm{ID}(R)$ be a fixed idempotent element. A proper ideal I of R with $e \notin I$ is said to be an e-prime ideal of R if whenever $x, y \in R$ with $xy \in I$, then $ex \in I$ or $ey \in I$. Among other things, we show that I is an e-prime ideal of R if and only if R/I is an $\bar{e}$-domain. We also investigate e-prime ideals in direct product and polynomial rings.

Key words and phrases: e-prime ideal, prime ideal, idempotent element.
AMS (MOS) Subject Classifications: 13A15.
ISSN 1814-0432, 2025, https://future-in-tech.net

2 Properties of e-prime ideals

Definition 2.1. *Let R be a ring and $e \in ID(R)$ be a fixed idempotent element. A proper ideal I of R with $e \notin I$ is said to be an e-prime ideal of R if whenever $x, y \in R$ with $xy \in I$, then $ex \in I$ or $ey \in I$.*

It is easy to observe that every prime ideal is an e-prime ideal but the converse statement is not always true as the following example shows.

Example 2.2. *Let R be a ring and let I be a prime ideal of R. Note that $e = (0, 1) \in ID(R \times R)$ and $e \notin I \times I$. We have $(I \times I :_{R \times R} e) = R \times I$ which is a prime ideal of $R \times R$. Using the following Proposition 2.3, $I \times I$ is an e-prime ideal of $R \times R$ but clearly $I \times I$ is not a prime ideal of $R \times R$.*

Proposition 2.3. *Let R be a ring and let $e \in ID(R)$. An ideal I of R with $e \notin I$ is an e-prime ideal of R if and only $(I :_R e)$ is a prime ideal of R.*

Proof. ($\Rightarrow$) Let $x, y \in R$ such that $xy \in (I :_R e)$. Then, $exy \in I$ and so $e^2 x = ex \in I$ or $ey \in I$ since I is an e-prime ideal. So, $x \in (I :_R e)$ or $y \in (I :_R e)$ and so $(I :_R e)$ is a prime ideal of R. ($\Leftarrow$) Let $x, y \in R$ such that $xy \in I \subseteq (I :_R e)$. Therefore, $x \in (I :_R e)$ or $y \in (I :_R e)$ as $(I :_R e)$ is a prime ideal of R. Then, $ex \in I$ or $ey \in I$. Consequently, I is an e-prime ideal of R. $\square$

Theorem 2.4. *Let R be a ring, $e \in ID(R)$, and I be a proper ideal of R with $e \notin I$. Then, I is an e-prime ideal of R if and only if whenever P, Q are ideals of R with $PQ \subseteq I$, then $eP \subseteq I$ or $eQ \subseteq I$.*

Proof. ($\Rightarrow$) Let P, Q be ideals of R with $PQ \subseteq I$ and suppose that $eP \nsubseteq I$. Take $ep \in eP - I$ and let $x \in Q$. Then, $epx \in I$ with $e^2 p = ep \notin I$ so $ex \in I$ since I is an e-prime ideal. Hence, $eQ \subseteq I$. ($\Leftarrow$) Let $x, y \in R$ such that $xy \in I$. Then, $(Rx)(Ry) \subseteq I$ and so $e(Rx) = Rex \subseteq I$ or $e(Ry) = Rey \subseteq I$ thus $ex \in I$ or $ey \in I$. Consequently, I is an e-prime ideal of R. $\square$

Proposition 2.5. *Let R be a ring, $e \in ID(R)$, and let I be a proper ideal of R with $e \notin I$. The following statements hold:*
(1) If I is a prime ideal of R and J is an ideal of R with $e \in J$, then IJ is an e-prime ideal of R.
(2) Let $f : R \to S$ be a ring homomorphism, and let P be an $f(e)$-prime ideal of S. Then, $f^{-1}(P)$ is an e-prime ideal of R.

Proof. (1) Note that $e \notin IJ$. Let $x, y \in R$ such that $xy \in IJ$. Since I is a prime ideal of R and $xy \in IJ \subseteq I$, $x \in I$ or $y \in I$. Therefore, $ex \in IJ$ or $ey \in IJ$ since $e \in J$ hence IJ is an e-prime ideal of R.
(2) Note that $e \notin f^{-1}(P)$ since we are assuming $f(e) \notin P$. Let $x, y \in R$ such that $xy \in f^{-1}(P)$. Then, $f(x)f(y) \in P$ and so $f(ex) \in P$ or $f(ey) \in P$. Thus, $ex \in f^{-1}(P)$ or $ey \in f^{-1}(P)$. As a result, $f^{-1}(P)$ is an e-prime ideal of R. □

Let R be a ring and let $e \in \mathrm{ID}(R)$ be a non-zero idempotent. We say that R is an e-domain if whenever $a, b \in R$ with $ab = 0$, then $ea = 0$ or $eb = 0$. If I is and ideal of R and $x \in R$, we use $\bar{x}$ to denote that image of x in R/I.

Theorem 2.6. *Let R be a ring, $e \in ID(R)$, and let I be a proper ideal of R with $e \notin I$. Then, I is an e-prime ideal of R if and only if R/I is an $\bar{e}$-domain.*

Proof. ($\Rightarrow$) Suppose that I is an e-prime ideal of R and let $\bar{x}, \bar{y} \in R/I$ such that $\bar{x}\bar{y} = \bar{0}$. It follows that $xy \in I$ so $ex \in I$ or $ey \in I$ and hence $\bar{e}\bar{x} = \bar{0}$ or $\bar{e}\bar{y} = \bar{0}$ thus R/I is an $\bar{e}$-domain. ($\Leftarrow$) Suppose that R/I is an $\bar{e}$-domain and let $x, y \in R$ such that $xy \in I$. Then, $\bar{x}\bar{y} = \bar{0}$ and so $\bar{e}\bar{x} = \bar{0}$ or $\bar{e}\bar{y} = \bar{0}$. Therefore, $ex \in I$ or $ey \in I$. As a result, I is an e-prime ideal of R. □

Proposition 2.7. *Let R be a ring, $e \in ID(R)$, and let $I \subseteq P$ be proper ideals of R with $e \notin P$. Then, P is an e-prime ideal of R if and only if P/I is an $\bar{e}$-prime ideal of R/I.*

Proof. ($\Rightarrow$) Let $\bar{x}, \bar{y} \in R/I$ such that $\bar{x}\bar{y} \in P/I$. Thus, $xy \in P$ and so $ex \in P$ or $ey \in P$ since P is an e-prime ideal of R. Hence, $\bar{e}\bar{x} \in P/I$ or $\bar{e}\bar{y} \in P/I$ and therefore P/I is an $\bar{e}$-prime ideal of R/I.
($\Leftarrow$) Consider the canonical homomorphism $\pi : R \to R/I$ defined by $\pi(x) = \bar{x}$. Since $\pi^{-1}(P/I) = P$ and P/I is an $\bar{e}$-prime ideal of R/I by assumption, we conclude that P is an e-prime ideal of R by Proposition 2.5 (2). □

Theorem 2.8 (The avoidance property of e-prime ideals). *Let R be a ring, $e \in ID(R)$, and let I_i be e-prime ideals of R where $i \in \{1, \dots, n\}$. If P is an ideal of R with $P \subseteq \cup_{i=1}^{n} I_i$, then $eP \subseteq I_j$ for some $j \in \{1, \dots, n\}$.*

Proof. By Proposition 2.3, $(I_i :_R e)$ is a prime ideal of R for all $i \in \{1, \dots, n\}$. Moreover, $P \subseteq \cup_{i=1}^{n} I_i \subseteq \cup_{i=1}^{n} (I_i :_R e)$. By the prime avoidance lemma, we have $P \subseteq (I_j :_R e)$ for some $j \in \{1, \dots\}$ and so $eP \subseteq I_j$ as stated. □

Theorem 2.9. *Let R_i be a ring, $e_i \in ID(R_i)$, and let I_i be a proper ideal of R_i where $i = 1, 2$. Then, $I_1 \times I_2$ is an (e_1, e_2)-prime ideal of $R_1 \times R_2$ if and only if $e_1 \in I_1$ and I_2 is an e_2-prime ideal of R_2 or $e_2 \in I_2$ and I_1 is an e_1-prime ideal of R_1.*

Proof. $(\Rightarrow)$ Note that $(1, 0)(0, 1) \in I_1 \times I_2$ and so $(e_1, e_2)(1, 0) \in I_1 \times I_2$ or $(e_1, e_2)(0, 1) \in I_1 \times I_2$ because $I_1 \times I_2$ is an (e_1, e_2)-prime ideal of $R_1 \times R_2$. It follows that $e_1 \in I_1$ and $e_2 \notin I_2$ or $e_2 \in I_2$ and $e_1 \notin I_1$. Without loss of generality, we may suppose that $e_1 \in I_1$ and $e_2 \notin I_2$. Let $x, y \in R_2$ such that $xy \in I_2$. Then, $(0, x)(0, y) \in I_1 \times I_2$ and hence $(e_1, e_2)(0, x) \in I_1 \times I_2$ or $(e_1, e_2)(0, y) \in I_1 \times I_2$. Consequently, $e_2x \in I_2$ or $e_2y \in I_2$ and thus I_2 is an e_2-prime ideal of R_2.
$(\Leftarrow)$ Suppose that $e_1 \in I_1$ and I_2 is an e_2-prime ideal of R_2. Then, $(e_1, e_2) \notin I_1 \times I_2$. Let $(x, y), (x', y') \in R_1 \times R_2$ such that $(x, y)(x', y') \in I_1 \times I_2$. Then, $yy' \in I_2$ and so $e_2y \in I_2$ or $e_2y' \in I_2$. As a result, $(e_1, e_2)(x, y) \in I_1 \times I_2$ or $(e_1, e_2)(x', y') \in I_1 \times I_2$ so $I_1 \times I_2$ is an (e_1, e_2)-prime ideal of $R_1 \times R_2$. A similar argument proves the other case. $\square$

Lemma 2.10. *Let R be a ring, $b \in R$, and let I be an ideal of R. In the ring $R[x]$, we have $(I[x] :_{R[x]} b) = (I :_R b)[x]$.*

Proof. The inclusion $(I :_R b)[x] \subseteq (I[x] :_{R[x]} b)$ is evident. So we prove the other inclusion. Let $f(x) = \sum_{n=0}^{k} a_n x^n \in (I[x] :_{R[x]} b)$. Then, $bf = \sum_{n=0}^{k} ba_n x^n \in I[x]$ thus $ba_n \in I$ and so $a_n \in (I :_R b)$ for all $n \in \{1, \dots k\}$. This shows that $f(x) \in (I :_R b)[x]$ and so $(I[x] :_{R[x]} b) \subseteq (I :_R b)[x]$. $\square$

Theorem 2.11. *Let R be a ring, $e \in ID(R)$, and let I be a proper ideal of R with $e \notin I$. Then, I is an e-prime ideal of R if and only if $I[x]$ is an e-prime ideal of $R[x]$.*

Proof. $(\Rightarrow)$ By Proposition 2.3, it suffices to prove that $(I[x] :_{R[x]} e)$ is a prime ideal of $R[x]$. Since I is an e-prime ideal of R, $(I :_R e)$ is a prime ideal of R and so $(I :_R e)[x]$ is a prime ideal of $R[x]$. By Lemma 2.10 $(I[x] :_{R[x]} e)$ is a prime ideal of $R[x]$.
$(\Leftarrow)$ Let $a, b \in R$ such that $ab \in I$. Then, $ab \in I[x]$ and so $ea \in I[x]$ or $eb \in I[x]$ since $I[x]$ is an e-prime ideal of $R[x]$. Consequently, $ea \in I$ or $eb \in I$. Consequently, I is an e-prime ideal of R. $\square$

References

[1] D. D. Anderson, E. Smith, Weakly prime ideals, Houston J. Math., **29,** (2003), 831–840.

[2] A. Hamed, A. Malek, S-prime ideals of a commutative ring, Beitr. Algebra Geom., **61,** (2020), 533–542.

[3] A. Yassine, M. J. Nikmehr, R. Nikandish, On 1-absorbing prime ideals of commutative rings, J. Algebra Appl., **20,** (2021), 2150175.