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On the exponential Diophantine equation $37^x - 5^y = z^2$

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Abstract

In this paper, we show that $(0, 0, 0)$ and $(1, 0, 6)$ are the only two non-negative integer solutions of the exponential Diophantine equation $37^x - 5^y = z^2$.

1 Introduction

The search for solutions to the exponential Diophantine equation in the form $a^x - b^y = z^2$ has been going on for a long time. In 2013, Sroysang [3] showed that the exponential Diophantine equation $2^x + 37^y = z^2$ has the unique non-negative integer solution. In 2020, Burshtein [1] investigated all the solutions of the Diophantine equations $13^x - 5^y = z^2$ and $19^x - 5^y = z^2$. In 2021, Thongnak, Chuayjan and Kaewong [5] proved that the Diophantine equation $7^x - 5^y = z^2$ has a unique non-negative integer solution. In 2023 Tadee [4] studied non-negative integer solutions of the Diophantine equation $n^x - 5^y = z^2$ where $n \equiv 11 \pmod{20}$.

In this paper, we show that $(0, 0, 0)$ and $(1, 0, 6)$ are the only two solutions (x, y, z) for the exponential Diophantine equation $37^x - 5^y = z^2$ where x, y and z are non-negative integers.

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2 Preliminaries

The main method in proving our results is modular arithmetic. For properties of modular arithmetic that are utilized in deriving the results, we refer the reader to [2].

We use the following lemmas which are related to congruences.

Lemma 2.1. *If n is an integer, then* $n^2 \equiv 0, 1, 4 \pmod 5$.

Lemma 2.2. *If n is an even number, then* $n^2 \equiv 0, 4 \pmod 6$.

The following definition and lemma are also required in our study.

Definition 2.3. *Let a and n be relatively prime positive integers. Then the least positive integer x such that $a^x \equiv 1 \pmod n$ is called the* order of a modulo n. *We denote the order of a modulo n by $ord_n a$*

Example. $\mathrm{ord}_{11}37 = 5$ and $\mathrm{ord}_{25}37 = 20$.

Lemma 2.4. *Let a and n be relatively prime integers with $n > 0$. Then the positive integer k is a solution of the congruence $a^k \equiv 1 \pmod n$ if and only if $ord_n a | k$.*

3 Main Results

Theorem 3.1. *Let x, y and z be non-negative integers. The Diophantine equation*

$$37^x - 5^y = z^2 \tag{3.1}$$

has exactly the two solutions $(x, y, z) = (0, 0, 0)$ and $(1, 0, 6)$.

Proof. Let (x, y, z) be a non-negative integer solution of (3.1). Then obviously z is even. Since $37 \equiv 1 \pmod 6$ and $5 \equiv -1 \pmod 6$, the equation (3.1) becomes $1 - (-1)^y \equiv z^2 \pmod 6$ and consequently y is even by Lemma 2.2. Next, we divide the proof into the following two cases:

Case 1. x is odd. If $x = 1$, then (3.1) is $37 - 5^y = z^2$. Since $z^2 \geq 0$, we have $37 \geq 5^y$. This implies that $y \in \{0, 1, 2\}$. One can easily see that $y = 0$ and hence $z = 6$. Consequently, $(1, 0, 6)$ is a solution of (3.1).

Suppose that $x > 1$. Then there exists a positive integer k such that $x = 2k + 1$. Therefore, the equation (3.1) becomes $37^{2k+1} - 5^y = z^2$. It follows that $37^{2k+1} \equiv z^2 \pmod 5$. Now, we consider

$$37^{2k+1} \equiv 37(37^{2k}) \equiv 2(2^{2k}) \equiv 2(-1)^k \equiv \begin{cases} 3 & \text{if} \quad k \text{ is odd} \\ 2 & \text{if} \quad k \text{ is even} \end{cases} \pmod 5.$$

This implies that $z^2 \equiv 2, 3 \pmod 5$. This contradicts Lemma 2.1. From this we obtain $(x, y, z) = (1, 0, 6)$ is as a solution of (3.1).

Case 2. x is even. If $x = 0$, then $1 - 5^y = z^2$. Since $z^2 \geq 0$, it follows that $y = 0$ and so $z = 0$. Consequently, $(x, y, z) = (0, 0, 0)$ is a solution of (3.1).

Suppose that $x \geq 2$. Then there exists a positive integer k such that $x = 2k$. Thus (3.1) becomes $37^{2k} - 5^y = z^2$. Hence

$$5^y = 37^{2k} - z^2 = (37^k - z)(37^k + z).$$

Since 5 is a prime number, we have $37^k - z = 5^a$ and $37^k + z = 5^{y-a}$, where a is a non-negative integer. Then

$$2(37^k) = 5^a + 5^{y-a} = 5^a(1 + 5^{y-2a})$$

which implies that $a = 0$. Therefore, $2(37^k) = 1 + 5^y$. Since y is even, $y = 2l$ for some integer l. This yields

$$2(37^k) = 1 + 25^l. \tag{3.2}$$

Therefore, $2(37^k) \equiv 1 \pmod{25}$. From (3.2) and the fact that $37^9 \equiv 2 \pmod{25}$, we obtain $2(37^k) \equiv 37^{k+9} \equiv 1 \pmod{25}$. Since $\mathrm{ord}_{25}37 = 20$, by Lemma 2.4, $20|(k+9)$ and so $k \equiv 11 \pmod{20}$. Then there exists an integer m such that $k = 20m + 11$. Equation (3.2) then becomes

$$2(37^{20m+11}) = 1 + 25^l.$$

From the fact that $37^5 \equiv 1 \pmod{11}$, we get the useful congruences:

$$2(37^{20m+11}) \equiv 2(37)(37^{5(4m+2)}) \equiv 2(4)(1) \equiv 8 \pmod{11}. \tag{3.3}$$

However,

$$1 + 25^l \equiv \begin{cases} 4 & \text{if} \quad l = 5u + 1 \\ 10 & \text{if} \quad l = 5u + 2 \\ 6 & \text{if} \quad l = 5u + 3 \\ 5 & \text{if} \quad l = 5u + 4 \\ 2 & \text{if} \quad l = 5u \end{cases} \pmod{11}$$

contradicts (3.3). From this, we obtain $(x, y, z) = (0, 0, 0)$ as the solution of (3.1). This completes the proof. $\square$

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