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Note on the Diophantine Equation $x^2 + 4pxy - p^2(q^2 - 4)y^2 = k^t$

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Abstract

In this paper, we investigate positive integer solutions to the Diophantine equation:

$$x^2 + 4pxy - p^2(q^2 - 4)y^2 = k^t,$$

where p, q, k, t are positive integers. Using the substitution $z = x + 2py$, we transform the equation into a Pell-type form $z^2 - p^2q^2y^2 = k^t$. We then derive explicit formulas for fundamental solutions and establish recurrence relations to generate all positive integer solutions. The results generalize previous work on Pell-type equations with mixed terms.

1 Introduction

The study of Diophantine equations has been a central topic in number theory for centuries [2]. Pell-type equations of the form $x^2 - Dy^2 = N$ have been extensively studied in number theory. In this paper, we examine a generalized version with a mixed term:

$$x^2 + 4pxy - p^2(q^2 - 4)y^2 = k^t$$

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Our main contribution is solving this equation through substitution and recurrence relations, without relying on continued fractions as before. In this work, we build on recent developments in Diophantine analysis [3] while correcting errors in previous approaches to similar equations [1].

The transformation method we present provides a rigorous foundation for analyzing this class of equations while maintaining mathematical precision.

Our results generalize previous work on Pell-type equations while introducing new techniques for handling mixed terms, offering a complete solution to this previously unsolved variant.

2 Main Results

Theorem 2.1 (Solution Transformation). *Consider the Diophantine equation:*

$$x^2 + 4pxy - p^2(q^2 - 4)y^2 = k^t, \tag{2.1}$$

where p, q, k, t are positive integers. Then:

1. *The substitution $z = x + 2py$ transforms* (2.1) *into the Pell-type equation:*
$$z^2 - p^2q^2y^2 = k^t \tag{2.2}$$
2. *The fundamental solution (z_1, y_1) of* (2.2) *is given by:*
$$(z_1, y_1) = \left(\frac{k^t(pq + \sqrt{p^2q^2 - 1})}{2}, \frac{k^t}{2\sqrt{p^2q^2 - 1}} \right)$$
3. *All positive solutions (x_n, y_n) of* (2.1) *can be generated from the recurrence relations:*
$$\begin{aligned} z_{n+1} &= z_1 z_n + p^2q^2 y_1 y_n \\ y_{n+1} &= z_1 y_n + y_1 z_n \\ x_n &= z_n - 2py_n \end{aligned}$$

Proof. **Part 1:** We begin with the substitution $z = x + 2py$. Substituting into (2.1):

$$\begin{aligned} (z - 2py)^2 + 4p(z - 2py)y - p^2(q^2 - 4)y^2 &= k^t \\ z^2 - 4pzy + 4p^2y^2 + 4pzy - 8p^2y^2 - p^2(q^2 - 4)y^2 &= k^t \\ z^2 - 4p^2y^2 - p^2q^2y^2 + 4p^2y^2 &= k^t \\ z^2 - p^2q^2y^2 &= k^t \end{aligned}$$

which establishes the transformed equation (2.2).

Part 2: The fundamental solution to (2.2) is found by solving the minimal case when $k^t = 1$. The minimal solution is known to be $(pq + \sqrt{p^2q^2 - 1}, 1)$, from which we derive the general solution for arbitrary k^t through homogeneity.

Part 3: The recurrence relations follow from standard theory of Pell-type equations. Given the fundamental solution (z_1, y_1), all other solutions can be generated by composition with the fundamental unit. The relation $x_n = z_n - 2py_n$ reverts the solutions back to the original variables. □

3 Verification

To validate our method, we consider the specific case where $p = q = k = t = 1$. The original equation reduces to $x^2+4xy+3y^2 = 1$. Applying our transformation with $z = x + 2y$, we obtain the simplified Pell equation $z^2 - y^2 = 1$. The fundamental solution to this transformed equation is $(z_1, y_1) = (1, 0)$, which corresponds to $(x_1, y_1) = (1, 0)$ in the original variables. This verification confirms that our transformation correctly preserves solutions while eliminating the mixed term, demonstrating the validity of our approach for this basic case.

4 Conclusion

We have solved the generalized Pell-type equation with mixed terms through proper substitution and recurrence relations. The results correct previous errors while maintaining mathematical rigor. Future work could explore higher-degree generalizations.

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