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Sum and Difference of Independent Topp-Leone Variables

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Abstract

The Topp-Leone distribution is quite captivating due to its finite support and multiple shapes for varying parameter values. In this article, we derive density functions of the sum and the difference of two independent random variables each distributed as Topp-Leone. We express these densities in terms of Appell's first hypergeometric function F_1 and Gauss hypergeometric function ${}_2F_1$.

1 Introduction

A random variable X is said to have a Topp-Leone distribution, denoted by $X \sim \mathrm{TL}(\nu;\sigma)$, if its pdf is given by

$$f_{\mathrm{TL}}(x;\nu,\sigma) = \frac{2\nu}{\sigma}\left(\frac{x}{\sigma}\right)^{\nu-1}\left(1-\frac{x}{\sigma}\right)\left(2-\frac{x}{\sigma}\right)^{\nu-1}, \quad 0 < x < \sigma < \infty. \qquad (1)$$

For $0 < \nu < 1$, the distribution defined by the density (1) is referred to as the J-shaped distribution by Topp and Leone [9] (also see Nadarajah and

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Kotz [4]). For $\nu > 1$, (1) attains different shapes depending on values of parameters (see Kotz and van Dorp [1]). For $\sigma = 1$, the density in (1) reduces to the standard Topp-Leone density and in this case we will write $X \sim \mathrm{TL}(\nu)$.

Recent years have seen the emergence of numerous research articles that concentrate on different facets of the Topp-Leone distribution, indicating that the Top-Leone family of probability distributions has attracted renewed attention (see Nagar, Zarrazola, and Echeverri-Valencia [6] and references cited therein). For more recent work, the reader is referred to Saini, Tomer and Garg [7]. The distributions of the sum and difference of Top-Leone variables, however, have received comparatively little attention.

In this article, we give distributions of sum and difference of two independent random variables having Topp-Leone and beta distributions.

2 Appell's first hypergeometric function

Appell's first hypergeometric function is defined in a series form as

$$F_1(a, b_1, b_2; c; z_1, z_2) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \frac{(a)_{i_1+i_2} (b_1)_{i_1} (b_2)_{i_2}}{(c)_{i_1+i_2}} \frac{z_1^{i_1} z_2^{i_2}}{i_1!\, i_2!}, \tag{1}$$

where $|z_1| < 1$ and $|z_2| < 1$ and the Pochhammer symbol $(a)_n$ is defined by $(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \ldots,$ with $(a)_0 = 1$. From (1), it follows that

$$\begin{aligned} F_1(a, b_1, b_2; c; z_1, z_2) &= \sum_{i_1=0}^{\infty} \frac{(a)_{i_1} (b_1)_{i_1}}{(c)_{i_1}} \frac{z_1^{i_1}}{i_1!}\ {}_2F_1(a + i_1, b_2; c + i_1; z_2) \\ &= \sum_{i_2=0}^{\infty} \frac{(a)_{i_2} (b_2)_{i_2}}{(c)_{i_2}} \frac{z_2^{i_2}}{i_2!}\ {}_2F_1(a + i_2, b_1; c + i_2; z_1), \end{aligned}$$

where ${}_2F_1$ is the Gauss hypergeometric function (Luke [2]). The integral representation of the Gauss hypergeometric function is given as (Luke [2, Eq. 3.6(1)]),

$${}_2F_1(a, b; c; z) = \frac{1}{B(a, c-a)} \int_0^1 \frac{t^{a-1}(1-t)^{c-a-1}}{(1-zt)^b}\, \mathrm{d}t, \tag{2}$$

where $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$, $|\arg(1-z)| < \pi$. Expanding $(1-zt)^{-b}$, $|zt| < 1$, in (2) and integrating with respect to t, the series expansion for ${}_2F_1$ is derived as

$$ {}_2F_1(a,b;c;z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \frac{z^j}{j!}. $$

Further, writing

$$ \frac{(a)_{i_1+i_2}}{(c)_{i_1+i_2}} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 v^{a+i_1+i_2-1}(1-v)^{c-a-1}\,\mathrm{d}v, \quad \mathrm{Re}(c) > \mathrm{Re}(a) > 0 $$

and

$$ \sum_{i_j=0}^{\infty} \frac{(b_j)_{i_j}(vz_j)^{i_j}}{i_j!} = (1-vz_j)^{-b_j}, \quad |vz_j| < 1, \quad j = 1,2 $$

in (1), one can derive an integral representation of Appell's first hypergeometric function of as

$$ F_1(a,b_1,b_2;c;z_1,z_2) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{v^{a-1}(1-v)^{c-a-1}\,\mathrm{d}v}{(1-vz_1)^{b_1}(1-vz_2)^{b_2}}, \tag{3} $$

where $|z_1| < 1$, $|z_2| < 1$, $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$. For further results the reader is referred to Srivastava and Karlsson [8].

3 Densities of sum and difference

In this section, we give densities of the sum and difference of two independent random variables each having Topp-Leone distribution.

Theorem 3.1. *Let X_1 and X_2 be independent random variables, $X_i \sim \mathrm{TL}(\nu_i)$, $i = 1,2$. Then, the pdf of $S = X_1 + X_2$ is derived as*

$$ \begin{aligned} f_S(s) &= 2^{\nu_1+1}\nu_1\nu_2 s^{\nu_1+\nu_2-1}(2-s)^{\nu_2-1}\frac{\Gamma(\nu_1)\Gamma(\nu_2)}{\Gamma(\nu_1+\nu_2)} \\ &\quad\times\left[(1-s)F_1\left(\nu_1,-\nu_1+1,-\nu_2+1;\nu_1+\nu_2;\frac{s}{2},-\frac{s}{2-s}\right)\right. \\ &\quad\left.+\frac{\nu_1\nu_2 s^2}{\nu_1+\nu_2+1}F_1\left(\nu_1+1,-\nu_1+1,-\nu_2+1;\nu_1+\nu_2+2;\frac{s}{2},-\frac{s}{2-s}\right)\right] \end{aligned} $$

for $0 < s < 1$, and

$$ \begin{aligned} f_S(s) &= 4\nu_1\nu_2(2-s)^3(3-s)^{\nu_1-1}\sum_{h=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-\nu_1+1)_h(-\nu_2+1)_i}{h!\,i!}(2-s)^{h+2i} \\ &\quad \frac{\Gamma(2i+2)\Gamma(h+2)}{\Gamma(2i+h+4)}{}_2F_1\left(2i+2,-\nu_1+1;2i+h+4;\frac{2-s}{3-s}\right) \end{aligned} $$

for $1 < s < 2$.

Proof. From Nadarajah [3, p. 90] and Nagar and Ramirez-Vanegas [5, Theorem 2.1], the density of S is given by

$$f_S(s) = \begin{cases} \int_0^s f_1(t) f_2(s-t)\,\mathrm{d}t, & \text{for} \quad 0 < s < 1,\\ \int_0^{2-s} f_1(s-1+t) f_2(1-t)\,\mathrm{d}t, & \text{for} \quad 1 < s < 2, \end{cases}$$

and therefore in our derivation we consider cases $0 < s < 1$ and $1 < s < 2$ separately. Using (1), the density of S, for $0 < s < 1$, is derived as

$$\begin{aligned} f_S(s) &= \int_0^s f_1(t) f_2(s-t)\,\mathrm{d}t \\ &= s \int_0^1 f_1(sw) f_2(s(1-w))\,\mathrm{d}w \\ &= 2^{\nu_1+1}\nu_1\nu_2 s^{\nu_1+\nu_2-1}(2-s)^{\nu_2-1}\int_0^1 \left[1-s+s^2 w(1-w)\right] \\ &\qquad w^{\nu_1-1}(1-w)^{\nu_2-1}\left(1-\frac{sw}{2}\right)^{\nu_1-1}\left(1+\frac{sw}{2-w}\right)^{\nu_2-1}\mathrm{d}w. \end{aligned}$$

Now, applying (3), the integral in the above density is evaluated in terms of Appell's first hypergeometric function and we get the desired result. Similarly, the density of S, for $1 < s < 2$, is obtained as

$$\begin{aligned} f_S(s) &= \int_0^{2-s} f_1(s-1+t) f_2(1-t)\,\mathrm{d}t \\ &= (2-s)\int_0^1 f_1(s-1+(2-s)w) f_2(1-(2-s)w)\,\mathrm{d}w \\ &= 4\nu_1\nu_2(2-s)^3(3-s)^{\nu_1-1}\int_0^1 w(1-w)\left[s-1+(2-s)w\right]^{\nu_1-1} \\ &\qquad [1-(2-s)^2w^2]^{\nu_2-1}[1+(2-s)w]^{\nu_2-1}\left[1-\frac{(2-s)w}{3-s}\right]^{\nu_1-1}\mathrm{d}w. \end{aligned}$$

Replacing $[s-1+(2-s)w]^{\nu_1-1}$ and $[1-(2-s)^2w^2]^{\nu_2-1}$ by their power series expansions; namely,

$$[s-1+(2-s)w]^{\nu_1-1} = \sum_{h=0}^{\infty}\frac{(-\nu_1+1)_h}{h!}(2-s)^h(1-w)^h$$

and

$$[1-(2-s)^2w^2]^{\nu_2-1} = \sum_{i=0}^{\infty}\frac{(-\nu_2+1)_i}{i!}(2-s)^{2i}w^{2i}$$

in the above integral and applying (2), one arrives at the desired result. $\square$

Theorem 3.2. *Let X_1 and X_2 be independent random variables, $X_i \sim \mathrm{TL}(\nu_i)$, $i = 1, 2$. Then, the pdf of $D = X_1 - X_2$ is obtained as*

$$
\begin{aligned}
f_D(d) = {} & 2^{\nu_1+1}\nu_1\nu_2(1+d)^{\nu_1+1}(2+d)^{\nu_2-1}\sum_{n=0}^{\infty}\frac{(-\nu_2+1)_n}{n!}(1+d)^n \\
& \frac{\Gamma(\nu_1)\Gamma(n+2)}{\Gamma(\nu_1+n+2)}\left[F_1\left(\nu_1, -\nu_1+1, -\nu_2+1; \nu_1+n+2; \frac{1+d}{2}, \frac{1+d}{2+d}\right)\right. \\
& \left. - \frac{\nu_1(1+d)}{\nu_1+n+2}F_1\left(\nu_1+1, -\nu_1+1, -\nu_2+1; \nu_1+n+3; \frac{1+d}{2}, \frac{1+d}{2+d}\right)\right]
\end{aligned}
$$

for $-1 < d < 0$, and

$$
\begin{aligned}
f_D(d) = {} & 2^{\nu_2+1}\nu_1\nu_2(1-d)^{\nu_2+1}(2-d)^{\nu_1-1}\sum_{m=0}^{\infty}\frac{(-\nu_1+1)_m}{m!}(1-d)^m \\
& \frac{\Gamma(\nu_2)\Gamma(m+2)}{\Gamma(\nu_2+m+2)}\left[F_1\left(\nu_2, -\nu_1+1, -\nu_2+1; \nu_2+m+2; \frac{1-d}{2-d}, \frac{1-d}{2}\right)\right. \\
& \left. - \frac{\nu_2(1-d)}{\nu_2+m+2}F_1\left(\nu_2+1, 1-\nu_1, 1-\nu_2; \nu_2+m+3; \frac{1-d}{2-d}, \frac{1-d}{d}\right)\right]
\end{aligned}
$$

for $0 < d < 1$.

Proof. Applying Nadarajah [3, p. 90] and Nagar and Ramirez-Vanegas [5, Theorem 2.1], the density of D is given as

$$
f_D(d) = \begin{cases} \int_0^{1+d} f_1(t) f_2(t-d)\,\mathrm{d}t, & \text{for} \quad -1 < d < 0, \\[2ex] \int_0^{1-d} f_1(d+t) f_2(t)\,\mathrm{d}t, & \text{for} \quad 0 < d < 1, \end{cases}
$$

and therefore we consider cases $-1 < d < 0$ and $0 < d < 1$ separately. Using the Topp-Leone density given in (1), the pdf of D, for $-1 < d < 0$, is derived as

$$
\begin{aligned}
f_D(d) &= \int_0^{1+d} f_1(t) f_2(t-d)\,\mathrm{d}t \\
&= (1+d)\int_0^1 f_1((1+d)w) f_2((1+d)w-d)\,\mathrm{d}w \\
&= 2^{\nu_1+1}\nu_1\nu_2(1+d)^{\nu_1+1}(2+d)^{\nu_2-1}\int_0^1 w^{\nu_1-1}(1-w)[1-(1+d)w] \\
&\quad [1-(1+d)(1-w)]^{\nu_2-1}\left[1-\frac{(1+d)w}{2}\right]^{\nu_1-1}\left[1-\frac{(1+d)w}{2+d}\right]^{\nu_2-1}\mathrm{d}w.
\end{aligned}
$$

Using the power series expansion

$$[1-(1+d)(1-w)]^{\nu_2-1}=\sum_{n=0}^{\infty}\frac{(-\nu_2+1)_n}{n!}(1+d)^n(1-w)^n$$

and applying (3), the above integral is evaluated in terms of Appell's hypergeometric function and we get the desired result. Using (1), the density of D, for $0<d<1$, is derived as

$$\begin{aligned}
f_D(d) &= \int_0^{1-d} f_1(d+t)f_2(t)\mathrm{d}t \\
&= (1-d)\int_0^1 f_1(d+(1-d)w)f_2((1-d)w)\,\mathrm{d}w \\
&= 2^{\nu_2+1}\nu_1\nu_2(1-d)^{\nu_2+1}(2-d)^{\nu_1-1}\int_0^1 w^{\nu_2-1}(1-w)[1-(1-d)w] \\
&\quad [1-(1-d)(1-w)]^{\nu_1-1}\left[1-\frac{(1-d)w}{2-d}\right]^{\nu_1-1}\left[1-\frac{(1-d)w}{2}\right]^{\nu_2-1}\mathrm{d}w.
\end{aligned}$$

Now, expanding $[1-(1-d)(1-w)]^{\nu_1-1}$ in power series,

$$[1-(1-d)(1-w)]^{\nu_1-1}=\sum_{m=0}^{\infty}\frac{(-\nu_1+1)_m}{m!}(1-d)^m(1-w)^m$$

and evaluating the above integral by using (3), we get the desired expression for the density. □

Finally, in Theorem 3.3 and Theorem 3.4, we give densities of X_1+X_2 and X_1-X_2 where X_1 and X_2 are independent, $X_1\sim\mathrm{TL}(\nu)$ and $X_2\sim\mathrm{B1}(a,b)$. A random variable X is said to have a beta type 1 distribution with parameters (a,b), $a>0$, $b>0$, denoted as $X\sim\mathrm{B1}(a,b)$, if its p.d.f. is given by

$$\{B(a,b)\}^{-1}x^{a-1}(1-x)^{b-1},\quad 0<x<1,$$

where $B(\alpha,\beta)=\Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$.

The proofs of these theorem are similar to those of Theorem 3.1 and Theorem 3.2 and can be constructed easily.

Theorem 3.3. *Let X_1 and X_2 be independent random variables, $X_1\sim\mathrm{TL}(\nu)$ and $X_2\sim\mathrm{B1}(a,b)$. Then, the pdf of $S=X_1+X_2$ is obtained as*

$$\begin{aligned}
&\frac{2^{\nu}\nu}{B(a,b)}s^{\nu+a-1}\sum_{m=0}^{\infty}\frac{(-b+1)_m}{m!}B(\nu,a+m)s^m\left[{}_2F_1\left(\nu,-\nu+1;\nu+m;\frac{s}{2}\right)\right.\\
&\quad\left.-\frac{\nu}{\nu+a+m}s\,{}_2F_1\left(\nu+1,-\nu+1;\nu+m+1;\frac{s}{2}\right)\right],\quad 0<s<1
\end{aligned}$$

and

$$\frac{2\nu}{B(a,b)}(2-s)^{b+1}(3-s)^{\nu-1}\sum_{h=0}^{\infty}\frac{(-\nu+1)_h}{h!}(2-s)^h\frac{\Gamma(b)\Gamma(h+2)}{\Gamma(b+h+2)}$$
$$\times F_1\left(b,-\nu+1,-\alpha+1;b+h+2;-\frac{2-s}{3-s},2-s\right),\quad 1<s<2.$$

Theorem 3.4. *Let X_1 and X_2 be independent random variables, $X_1 \sim \mathrm{TL}(\nu)$ and $X_2 \sim \mathrm{B1}(a,b)$. Then, the pdf of $D = X_1 - X_2$ is derived as*

$$\frac{2^\nu\nu}{B(a,b)}(1+d)^{\nu+b-1}\sum_{m=0}^{\infty}\frac{(-\nu+1)_m}{m!}(1+d)^m\frac{\Gamma(\nu)\Gamma(b+m)}{\Gamma(\nu+b+m)}$$
$$\times\left[{}_2F_1\left(\nu,-\nu+1;\nu+b+m;\frac{1+d}{2}\right)\right.$$
$$\left.-\frac{\nu(1+d)}{\nu+b+m}{}_2F_1\left(\nu+1,-\nu+1;\nu+b+m+1;\frac{1+d}{2}\right)\right],\quad -1<d<0$$

and

$$\frac{2\nu}{B(a,b)}(1-d)^{a+1}(2-d)^{\nu-1}\sum_{m=0}^{\infty}\frac{(-\nu+1)_m}{m!}(1-d)^m\frac{\Gamma(a)\Gamma(m+2)}{\Gamma(a+m+2)}$$
$$\times\ F_1\left(a,-\nu+1,-b+1;a+m+2;\frac{1-d}{2-d},1-d\right),\quad 0<d<1.$$

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