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On the Diophantine equation $a^x + b^y = z^2$ where $a \equiv 1 \pmod 3$ and $b \equiv 1 \pmod 3$

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Abstract

In this article, we show that the Diophantine equation $a^x + b^y = z^2$ has no non-negative integer solution, where a, b are positive odd integers with $a \equiv 1 \pmod 3$, $b \equiv 1 \pmod 3$, and x, y, z are non-negative integers.

1 Introduction

Diophantine equations are equations in which the solutions, are required to be integers. It is a popular topic in Number theory. The Diophantine equations of the form $a^x + b^y = z^2$, where a and b are fixed integers have been studied by many mathematicians (see for instance [1, 2, 3, 4]) and the references therein. In 2020, Dokchann and Pakapongpun [2] showed that the Diophantine equation $a^x + (a + 2)^y = z^2$, where a is a positive integer with $a \equiv 5 \pmod{42}$, has no non-negative integer solution. In 2022, N. Viriyapong and C. Viriyapong [4] showed that the Diophantine equation $n^x + 19^y = z^2$ has only one solution $(n, x, y, z) = (2, 3, 0, 3)$, where n is a positive integer with $n \equiv 2 \pmod{57}$ and x, y, z are non-negative integers.

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In this paper, we investigate the existence of a solution of the Diophantine equation $a^x+b^y=z^2$, where a, b are positive odd integers with $a \equiv 1 \pmod 3$, $b \equiv 1 \pmod 3$, and x, y, z are non-negative integers.

2 Preliminaries

We begin this section with the following lemma, which we will use to prove the main theorem.

Lemma 2.1. *Let a be a positive odd integer such that $a \equiv 1 \pmod 3$. Then the Diophantine equation $a^x + 1 = z^2$ has no non-negative integer solution.*

Proof. Suppose that there exist non-negative integers x and z such that $a^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$, which is a contradiction. Thus, $x \geq 1$. Since a is an odd number, we obtain a^x is an odd number, for all $x \geq 1$, which implies that $z^2 = a^x + 1$ is an even number. Thus, $z^2 \equiv 0 \pmod 3$ or $z^2 \equiv 1 \pmod 3$. Since $a \equiv 1 \pmod 3$, we obtain $a^x \equiv 1 \pmod 3$, for all $x \geq 1$. Thus, $a^x + 1 \equiv 2 \pmod 3$. That is, $z^2 \equiv 2 \pmod 3$, which is a contradiction. Therefore, the Diophantine equation $a^x + 1 = z^2$ has no non-negative integer solution. □

3 Main results

Theorem 3.1. *Let a and b be any positive odd integers such that $a \equiv 1 \pmod 3$ and $b \equiv 1 \pmod 3$. Then, the Diophantine equation $a^x + b^y = z^2$ has no non-negative integer solutions.*

Proof. Assume that there exist non-negative integers x, y and z such that $a^x + b^y = z^2$. If $x = 0$ and $y = 0$, then $z^2 = 2$, which is a contradiction. If ($x = 0$ and $y > 0$) or ($x > 0$ and $y = 0$), then it follows from Lemma 2.1 that $a^x + b^y = z^2$ has no non-negative integer solution. Thus, $x \geq 1$ and $y \geq 1$. Since a and b are odd numbers, we obtain a^x and b^y are odd numbers for all $x, y \in \mathbb{Z}^+$. Thus, $z^2 = a^x + b^y$ is an even number which implies that $z^2 \equiv 0 \pmod 3$ or $z^2 \equiv 1 \pmod 3$. Now, since $a \equiv 1 \pmod 3$ and $b \equiv 1 \pmod 3$, we get that $a^x \equiv 1 \pmod 3$ and $b^y \equiv 1 \pmod 3$, for all $x, y \in \mathbb{Z}^+$. This implies that $a^x + b^y \equiv 2 \pmod 3$. Thus, $z^2 \equiv 2 \pmod 3$, which is a contradiction. Therefore, the Diophantine equation $a^x + b^y = z^2$ has no non-negative integer solutions. □

Finally, we shall apply Theorem 3.1 when $a = 19$, $b = 31$ and $a = 25$, $b = 115$, respectively.

Corollary 3.2. *The Diophantine equation $19^x + 31^y = z^2$ has no non-negative integer solutions, where x, y and z are non-negative integers.*

Proof. Since 19 and 31 are positive odd integers such that $19 \equiv 1 \pmod 3$ and $31 \equiv 1 \pmod 3$, by Theorem 3.1, the Diophantine equation $19^x + 31^y = z^2$ has no non-negative integer solutions. $\square$

Corollary 3.3. *The Diophantine equation $25^x + 115^y = z^2$ has no non-negative integer solution, where x, y and z are non-negative integers.*

Proof. Since 25 and 115 are positive odd integers such that $25 \equiv 1 \pmod 3$ and $115 \equiv 1 \pmod 3$, by Theorem 3.1, the Diophantine equation $25^x + 115^y = z^2$ has no non-negative integer solutions. $\square$

4 Conclusion

In this article, we showed that the Diophantine equation $a^x + b^y = z^2$, where a, b are positive odd integers with $a \equiv 1 \pmod 3, b \equiv 1 \pmod 3$, and x, y, z are non-negative integers has no non-negative integer solutions. The above result can apply to the case where a, b are prime or composite as can be easily seen from Corollaries 3.2 and 3.3.

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