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# Some identities for Gibonacci Sequence

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**Abstract**

In this paper, we employ matrix methods to analyze the Gibonacci sequence, a generalization of Fibonacci-Lucas sequences. We derive an explicit formula for the $nth$ term of this sequence and delve into its fundamental properties.

## 1 Introduction

The Fibonacci and Lucas sequences are two of the most famous number sequences in mathematics, captivating mathematicians and enthusiasts for centuries. These sequences possess intriguing properties and have applications across various fields from nature to computer science.

The Fibonacci sequence, denoted as $\{F_n\}_{n=0}^{\infty}$, is defined by the recurrence relation: $F_n = F_{n-1} + F_{n-2}$, for all $n \geqslant 2$, with initial values $F_0 = 0$ and $F_1 = 1$. The Lucas sequence, denoted as $\{L_n\}_{n=0}^{\infty}$, is closely related to the Fibonacci sequence and is defined by the same recurrence relation but with different initial conditions: $L_0 = 2$ and $L_1 = 1$.

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Both sequences exhibit a fascinating relationship with linear algebra. Koshy [4] explored the use of matrices and determinants in Fibonacci numbers. Cahill and Narayan [1] revealed how Fibonacci and Lucas numbers can be represented as determinants of some tridiagonal matrices. Macfarlane [5] extended this approach, utilizing determinant properties to derive new identities involving Fibonacci and related numbers. Additionally, numerous generalizations of these numbers have been presented in various forms [2],[3],[6].

In this paper, we investigate a generalized Fibonacci-Lucas type sequence, known as the Gibonacci sequence, using a matrix-based approach. We derive an explicit formula for the general term of these sequences and explore some of its fundamental properties using matrix methods.

## 2 Main results

The Gibonacci sequence $\{G_n\}_{n=0}^{\infty}$ is defined by the recurrence relation:

$$G_n = G_{n-1} + G_{n-2}, \text{ for all } n \geqslant 2,$$

with arbitrary initial values $G_0 = a$ and $G_1 = b$, where $a, b$ are integers.
The first few terms of $\{G_n\}_{n=0}^{\infty}$ are $a, b, a+b, a+2b, 2a+3b, ....$ Each number in the sequence is called a Gibonacci number, with $G_n$ denoting the $n$th term. Notably, if $a = 0$ and $b = 1$, then the sequence reduces to the classical Fibonacci sequence. Moreover, if $a = 2$ and $b = 1$, then it becomes the classical Lucas sequence.
We now analyze the Gibonacci sequence using eigenvalues and eigenvectors of a $2 \times 2$ matrix and find a general explicit formula for its $n$th term $\{G_n\}_{n=0}^{\infty}$.
Let us consider the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and the matrix $X_n = \begin{bmatrix} G_n \\ G_{n+1} \end{bmatrix}$, associated with the Gibonacci sequence $\{G_n\}_{n=0}^{\infty}$.
It is easy to show that $AX_n = X_{n+1}$ and $X_n = A^n X_0$, where $X_0 = \begin{bmatrix} a \\ b \end{bmatrix}$.

**Theorem 2.1.** *The nth Gibonacci sequence is given by for nonnegative integer n,*

$$G_n = \frac{a\left(\alpha^{n-1} - \beta^{n-1}\right) + b(\alpha^n - \beta^n)}{\sqrt{5}},$$

*where $\alpha, \beta$ are the roots of the characteristic equation of the matrix* $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

**Proof.** The characteristic equation of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ is $\lambda^2 - \lambda - 1 = 0$. So, the eigenvalues are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.
All eigenvectors $v$ corresponding to $\alpha = \frac{1+\sqrt{5}}{2}$ must satisfy $\begin{bmatrix} -\alpha & 1 \\ 1 & 1-\alpha \end{bmatrix} v = 0$ and so we may take $v_1 = \begin{bmatrix} -\beta \\ 1 \end{bmatrix}$. Similarly, all eigenvectors $u$ corresponding to $\beta = \frac{1-\sqrt{5}}{2}$ must satisfy $\begin{bmatrix} -\beta & 1 \\ 1 & 1-\beta \end{bmatrix} u = 0$ and so we may take $u_1 = \begin{bmatrix} -\alpha \\ 1 \end{bmatrix}$.
Let $P = \begin{bmatrix} -\beta & -\alpha \\ 1 & 1 \end{bmatrix}$. Then,
$P^{-1}AP = \frac{1}{\alpha-\beta} \begin{bmatrix} 1 & \alpha \\ -1 & -\beta \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -\beta & -\alpha \\ 1 & 1 \end{bmatrix}$
$= \frac{1}{\alpha-\beta} \begin{bmatrix} \alpha(\alpha-\beta) & 0 \\ 0 & \beta(\alpha-\beta) \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$
It follows that
$A^n = P \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} P^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha^{n-1} - \beta^{n-1} & \alpha^n - \beta^n \\ \alpha^n - \beta^n & \alpha^{n+1} - \beta^{n+1} \end{bmatrix}$
Thus, $\begin{bmatrix} G_n \\ G_{n+1} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha^{n-1} - \beta^{n-1} & \alpha^n - \beta^n \\ \alpha^n - \beta^n & \alpha^{n+1} - \beta^{n+1} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$.
By equating the corresponding entries of the matrices on both sides, we obtain $G_n = \frac{a\left(\alpha^{n-1}-\beta^{n-1}\right)+b(\alpha^n-\beta^n)}{\sqrt{5}}$ and so the proof of the theorem is complete.□

**Theorem 2.2.** *For a nonnegative integer n, we have the following formula:*

$$G_0^2 + G_1^2 + G_2^2 + \cdots + G_n^2 = G_n G_{n+1} + a^2 - ab.$$

**Proof.** The proof by induction on $n$. The base case of $n = 0$ is obvious. Now, assume the theorem is true for some $n \geq 0$. Then

$$\begin{aligned} G_0^2 + G_1^2 + G_2^2 + \cdots + G_n^2 + G_{n+1}^2 &= \left(G_n G_{n+1} + a^2 - ab\right) + G_{n+1}^2 \\ &= \left(G_n G_{n+1} + G_{n+1}^2\right) + a^2 - ab \\ &= G_{n+1}\left(G_n + G_{n+1}\right) + a^2 - ab \\ &= G_{n+1} G_{n+2} + a^2 - ab \end{aligned}$$

This is the statement of the theorem for $n+1$. Therefore, the result follows.□
From now on, let $S_n$ represent the sum of first $n+1$ terms of the Gibonacci sequence $\{G_n\}_{n=0}^{\alpha}$.

**Theorem 2.3.** *Let $n \geq 2$ be an integer. Then $S_n = b + S_{n-1} + S_{n-2}$.*

The proof can be easily demonstrated using induction on $n$.

**Corollary 2.4.** *Let $n \geq 0$ be an integer. Then $S_n = G_{n+2} - b$.*

From this result, we derive the well-known identities involving Fibonacci and Lucas numbers.

**Corollary 2.5.** *Let $n \geq 0$ be an integer. Then $\sum_{k=0}^{n} F_k = F_{n+2} - 1$ and $\sum_{k=0}^{n} L_k = L_{n+2} - 1$.*

Next, we consider the matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and the matrix

$$Y_n = \begin{bmatrix} 2b & 2b & 2b \\ S_n - b & S_{n+1} - b & S_{n+2} - b \\ S_{n+1} - b & S_{n+2} - b & S_{n+3} - b \end{bmatrix} \text{ for any integer } n \geq 0.$$

It is straightforward to show that for any integer $n \geq 0, BY_n = Y_{n+1}$ and $Y_n = B^n Y_0$,
where $Y_0 = \begin{bmatrix} 2b & 2b & 2b \\ a - b & a & 2a + b \\ a & 2a + b & 3a + 3b \end{bmatrix}$. This generator naturally leads to the Cassini formula for the Gibonacci numbers.

**Theorem 2.6.** *Let $n \geq 0$ be an integer. Then*

$$b\left(G_{n+1}G_{n+3} - G_{n+2}^2\right) = (-1)^n b\left(b^2 - ab - a^2\right).$$

**Proof.** For $n = 0$, the result is straightforward.
Now, let $n \geq 1$ be an integer. Since $Y_n = B^n Y_0$, we have

$$\det Y_n = (\det B)^n \det Y_0.$$

By substituting $\det Y_n = 2b\left(G_{n+1}G_{n+3} - G_{n+2}^2\right), \det B = -1$ and $\det Y_0 = 2b\left(b^2 - ab - a^2\right)$, the desired result follows.□

From Theorem 2.6 we obtain the well-known Cassini formulas for Fibonacci numbers and Lucas numbers.

**Corollary 2.7.** *Let $n \geq 0$ be an integer. Then $F_{n+1}F_{n+3} - F_{n+2}^2 = (-1)^n$ and $L_{n+1}L_{n+3} - L_{n+2}^2 = 5(-1)^{n+1}$.*

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