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Zero-One Inflated Bell Distribution and Its Application to Insurance Data

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Abstract

The Bell distribution is a simple discrete distribution with one parameter. It has interesting properties, such as being a part of the one-parameter exponential family of distributions and being infinitely divisible. Moreover, the Bell distribution has a variance larger than the mean, indicating that it may be suitable for overdispersed data. With this performance, the Bell distribution is more useful than the Poisson distribution, which is the most popular model for count data. Inflated models have become quite popular in the recent applied statistical literature. In many scientific studies, we often experience situations in which the data consists of a large proportion of zeros and ones. To model count data with excess zeros and excess ones, this paper presents a zero-one inflated Bell distribution. Some properties of the zero-one inflated Bell distribution are also included, such as the probability mass function, probability generating function, moment about the origin, mean, and variance. In addition, in this paper,

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we investigate the parameter estimation of the zero-one inflated Bell distribution by using the maximum likelihood method. Finally, the insurance dataset is used to show how useful the zero-one inflated Bell distribution is in real life and to see how it compares to other well-known distributions in terms of fit.

1 Introduction

Undoubtedly, the one-parameter Poisson distribution is the most popular model for count data used in practice, mainly because of its simplicity. On the other hand, a major drawback of this distribution is that its variance is restricted to be equal to the mean. Empirically, we often find data that exhibit overdispersion, which is variance larger than the mean, and hence the one-parameter Poisson distribution may not be suitable in such a case. For this restriction, the negative binomial distribution and other developed extended Poisson distributions are introduced with two or three parameters.

The Bell distribution, introduced by Castellares et al. [1], is a simple discrete distribution with one parameter. It has many interesting properties, such as:
(i) it is a one-parameter distribution;
(ii) it belongs to the one-parameter exponential family of distributions;
(iii) the Poisson distribution is not nested in the Bell family, but for small values of the parameter, the Bell distribution approaches the Poisson distribution; and
(iv) it is infinitely divisible.

A discrete random variable X has a Bell distribution with a parameter if its probability mass function is given by

$$g(x) = \frac{\theta^x B_x \exp\left(1 - e^\theta\right)}{x!}, \ x = 0, 1, 2, 3, ... \tag{1.1}$$

where $\theta > 0$ and $B_x = \frac{1}{e}\sum_{k=0}^{\infty} \frac{k^x}{k!}$ are the Bell numbers starting with $B_0 = B_1 = 1$. The mean and variance of the Bell distribution are $E(X) = \theta\exp(\theta)$ and $\mathrm{Var}(X) = \theta(1+\theta)\exp(\theta)$, respectively.

Moreover, the Bell distribution stands out as a parsimonious model for handling overdispersion, requiring no additional parameters compared to other two- and three-parameter distributions. This simplicity enhances its appeal for modeling overdispersed data, making it both efficient and effective [2]. Another characteristic of overdispersed data may result from some occurrence, including excess zeros or both excess zero and one frequencies simultaneously. The fitting model for high zero frequency is called a zero-inflated model. The models have been developed to address this issue, in which the zero-inflated Poisson distribution is introduced by Lambert [3]. Lemonte et al. [2] proposed the zero-inflated Bell distribution.

Nevertheless, the model for fitting excess zeros and ones data is called a zero-one inflated model. Alshkaki [4] extended the zero-inflated power series distributions by introducing zero-one inflated models, which inflate both zero and one frequencies. This innovation broadens the scope of zero-inflated models, providing a more flexible framework. The zero-one inflated Poisson distribution and the zero-one inflated negative binomial distribution were proposed by Alshkaki [5] and Alshkaki [6], respectively. Furthermore, there are many related articles, such as zero-one inflated Poisson-Sushila [7], zero-one inflated negative binomial-Sushila distributions [8], zero-one inflated negative binomial-beta exponential distribution [9], and zero-one-inflated Poisson-Lindley distribution [10].

In insurance data, zero-inflated models and zero-one-inflated models are used to address datasets with an excess of zeros—a common scenario for claim data where many policyholders might not file any claims, which is common in areas like auto or health insurance. Zero-one inflated models extend zero-inflated models by also accounting for an overabundance of ones in the data, helpful when both zero and one counts are frequent. This two-part inflation can better show the range of outcomes in some insurance situations, like when policies have a lot of small claims (count of one) and a lot of zero claims.

As mentioned above, the Bell distribution is the one-parameter distribution that has both efficient and effective handling of overdispersion [2]. The goal of this work is to come up with a new model, the zero-one inflated Bell distribution, that builds on the Bell distribution and the zero-inflated Bell distribution. It has only three parameters. Some properties of the zero-one inflated Bell distribution are also derived. For parameter estimation, the maximum likelihood estimators are obtained. The performance of the proposed distribution in fitting insurance datasets with a large number of zeros and ones compared to a zero-to-one inflated Poisson distribution is shown in the application section.

2 Zero-One Inflated Bell Distribution

Let Y be a random variable from a zero-one inflated Bell distribution with parameters $\theta > 0$, $0 < \pi_0 < 1$, and $0 < \pi_1 < 1$, written as $Y \sim \mathrm{ZOIB}(\theta, \pi_0, \pi_1)$.

Theorem 2.1. *The probability mass function (pmf) of Y is given by*

$$f(y) = \begin{cases} \pi_0 + (1-\pi_0-\pi_1)\exp\left(1-e^\theta\right) & , y=0 \\ \pi_1 + (1-\pi_0-\pi_1)\,\theta\exp\left(1-e^\theta\right) & , y=1 \\ (1-\pi_0-\pi_1)\frac{\theta^y B_y \exp\left(1-e^\theta\right)}{y!} & , y \geq 2 \end{cases} \tag{2.2}$$

where $\theta > 0$, $0 < \pi_0 < 1$, and $0 < \pi_1 < 1$.

Proof. The zero–one inflated model is an extra proportion added to the proportion of zero, and then the pmf of zero-one inflated model is defined by

$$f(y) = \begin{cases} \pi_0 + (1-\pi_0-\pi_1)\,g(0) & y=0 \\ \pi_1 + (1-\pi_0-\pi_1)\,g(1) & y=1 \\ (1-\pi_0-\pi_1)\,g(y) & y=2,3,... \end{cases}$$

Let Y be a random variable from the Bell distribution with parameters $\theta > 0$. It will be shown that $g(0) = \exp\left(1-e^\theta\right)$, $g(1) = \theta\exp\left(1-e^\theta\right)$, and $g(y) = \frac{\theta^y B_y \exp\left(1-e^\theta\right)}{y!}$.

Corollary 2.2. *The $Y \sim ZOIB(\theta, \pi_0, \pi_1)$ reduces to the Bell distribution if $\pi_0 = 0$ and $\pi_1 = 0$.*

Corollary 2.3. *The $Y \sim ZOIB(\theta, \pi_0, \pi_1)$ reduces to the one inflated Bell distribution if $\pi_0 = 0$.*

Corollary 2.4. *The $Y \sim ZOIB(\theta, \pi_0, \pi_1)$ reduces to the zero inflated Bell distribution as proposed by Lemonte et al. [2] if $\pi_1 = 0$.*

3 Some Properties

This section delineates fundamental statistical characteristics of this distribution; namely, the probability generating function, the rth moment around the origin, as well as the mean and the variance.

3.1 Probability generating function

The probability generating function is given by

$$G_Y(s) = \pi_0 + \pi_1 s + (1-\pi_0-\pi_1)\exp\left(e^{s\theta} - e^\theta\right)$$

3.2 Moment

The rth moment about origin of $Y \sim \text{ZOIB}(\theta, \pi_0, \pi_1)$ is given by

$$E\left(Y^r\right) = \pi_1 + (1 - \pi_0 - \pi_1) \exp\left(1 - e^{\theta}\right) \sum_{y=1}^{\infty} \frac{y^r \theta^y B_y}{y!} \tag{3.3}$$

where $\theta > 0$, $0 < \pi_0 < 1$, and $0 < \pi_1 < 1$.

3.3 Mean

The first moment about the origin of $Y \sim \text{ZOIB}\left(\theta, \pi_0, \pi_1\right)$ is the mean of $Y \sim \text{ZOIB}\left(\theta, \pi_0, \pi_1\right)$;

$$E\left(Y\right) = \pi_1 + (1 - \pi_0 - \pi_1)\, \theta \exp(\theta)$$

Proof. The first moment about origin of $Y \sim \text{ZOIB}\left(\theta, \pi_0, \pi_1\right)$ in (3.3) will be

$$E\left(Y\right) = \pi_1 + (1 - \pi_0 - \pi_1) \exp\left(1 - e^{\theta}\right) \sum_{y=1}^{\infty} \frac{y \theta^y B_y}{y!}$$

and $\exp\left(1 - e^{\theta}\right) \sum_{y=1}^{\infty} \frac{y \theta^y B_y}{y!} = \theta e^{\theta}$ consists of the mean of the Bell distribution.

3.4 Variance

The variance of $Y \sim \text{ZOIB}\left(\theta, \pi_0, \pi_1\right)$ is

$$\begin{aligned} Var(Y) &= \pi_1 + (1 - \pi_0 - \pi_1)\, \theta \exp(\theta) \left(\theta \exp(\theta) + \theta + 1\right) \\ &\quad - \left[\pi_1 + (1 - \pi_0 - \pi_1)\, \theta \exp(\theta)\right]^2. \end{aligned}$$

Proof. The second moment about origin of $Y \sim \text{ZOIB}\left(\theta, \pi_0, \pi_1\right)$ in (3.3) will be

$$\begin{aligned} E\left(Y^2\right) - E\left(Y\right)^2 &= \pi_1 + (1 - \pi_0 - \pi_1) \sum_{y=1}^{\infty} \frac{y^2 e^{1-e^{\theta}} \theta^y B_y}{y!} \\ &\quad - \left[\pi_1 + (1 - \pi_0 - \pi_1)\, \theta \exp(\theta)\right]^2 \\ &= \pi_1 + (1 - \pi_0 - \pi_1)\, \theta e^{\theta} \left(\theta e^{\theta} + \theta + 1\right) \\ &\quad - \left[\pi_1 + (1 - \pi_0 - \pi_1)\, \theta \exp(\theta)\right]^2 \end{aligned}$$

and $\exp\left(1 - e^{\theta}\right) \sum_{y=1}^{\infty} \frac{y^2 \theta^y B_y}{y!} = \theta e^{\theta} \left(\theta e^{\theta} + \theta + 1\right)$ consists of the second moment of the Bell distribution.

4 Parameter Estimation

The maximum likelihood method is considered to estimate the parameters for the zero-one inflated Bell (ZOIB) distribution. The likelihood function of ZOIB(θ, π_0, π_1) is given by

$$
\begin{aligned}
L(\theta, \pi_0, \pi_1) = \prod_{i=1}^{n} & \left[\left(\pi_0 + (1 - \pi_0 - \pi_1) \exp\left(1 - e^{\theta}\right) \right)^{a_i} \right] \\
\times & \left[\left(\pi_1 + (1 - \pi_0 - \pi_1) \, \theta \exp\left(1 - e^{\theta}\right) \right)^{b_i} \right] \\
\times & \left[\left((1 - \pi_0 - \pi_1) \frac{\theta^{y_i} B_{y_i} \exp\left(1 - e^{\theta}\right)}{y_i!} \right)^{1 - a_i - b_i} \right]
\end{aligned}
\tag{4.4}
$$

where $a_i = \begin{cases} 1, & \text{if } y_i = 0 \\ 0, & \text{if } y_i \neq 0 \end{cases}$ and $b_i = \begin{cases} 1, & \text{if } y_i = 1 \\ 0, & \text{if } y_i \neq 1 \end{cases}$, with the corresponding log-likelihood function:

$$
\begin{aligned}
LL(\theta, \pi_0, \pi_1) = & \left(\sum_{i=1}^{n} a_i \right) \log\left(\pi_0 + (1 - \pi_0 - \pi_1) \exp\left(1 - e^{\theta}\right) \right) \\
& + \left(\sum_{i=1}^{n} b_i \right) \log\left(\pi_1 + (1 - \pi_0 - \pi_1) \, \theta \exp\left(1 - e^{\theta}\right) \right) \\
& + \sum_{i=1}^{n} (1 - a_i - b_i) \log\left(1 - \pi_0 - \pi_1\right) \\
& + \sum_{i=1}^{n} (1 - a_i - b_i) \, y_i \log(\theta) + \sum_{i=1}^{n} (1 - a_i - b_i) \log\left(B_{y_i}\right) \\
& + \sum_{i=1}^{n} (1 - a_i - b_i) \left(1 - e^{\theta}\right) - \sum_{i=1}^{n} (1 - a_i - b_i) \log(y_i!)
\end{aligned}
$$

Here, let $n_0 = \sum_{i=1}^{n} a_i$ be the number of zeros, $n_1 = \sum_{i=1}^{n} b_i$ be the number of ones, and $c_i = 1 - a_i - b_i$.

$$
\begin{aligned}
LL\left(\theta,\pi_0,\pi_1\right) &= n_0 \log\left(\pi_0 + \left(1-\pi_0-\pi_1\right)\exp\left(1-e^{\theta}\right)\right) \\
&+ n_1 \log\left(\pi_1 + \left(1-\pi_0-\pi_1\right)\theta\exp\left(1-e^{\theta}\right)\right) \\
&+ \left(n-n_0-n_1\right)\log\left(1-\pi_0-\pi_1\right) + \sum_{i=1}^{n} c_i y_i \log\left(\theta\right) \\
&+ \sum_{i=1}^{n} c_i \log\left(B_{y_i}\right) + \left(n-n_0-n_1\right)\left(1-e^{\theta}\right) - \sum_{i=1}^{n} c_i \log\left(y_i!\right)
\end{aligned}
$$

The optimal value of the parameters can be obtained by the first partial derivatives of the log-likelihood function with respect to θ, π_0, and π_1. Then, it gives rise to the following equations:

$$
\begin{aligned}
\frac{\partial LL}{\partial \theta} &= \frac{n_0\left(1-\pi_0-\pi_1\right)\exp\left(1-e^{\theta}\right)\left(-e^{\theta}\right)}{\pi_1 + \left(1-\pi_0-\pi_1\right)\exp\left(1-e^{\theta}\right)} + \sum_{i=1}^{n}\frac{c_i y_i}{\theta} - \left(n-n_0-n_1\right)e^{\theta} \\
&+ \frac{n_1\left(\pi_1 + \left(1-\pi_0-\pi_1\right)\exp\left(1-e^{\theta}\right) + \left(1-\pi_0-\pi_1\right)\theta\exp\left(1-e^{\theta}\right)\left(-e^{\theta}\right)\right)}{\pi_1 + \left(1-\pi_0-\pi_1\right)\theta\exp\left(1-e^{\theta}\right)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial LL}{\partial \pi_0} &= \frac{n_0\left(1-\exp\left(1-e^{\theta}\right)\right)}{\pi_0 + \exp\left(1-e^{\theta}\right)\left(1-\pi_0-\pi_1\right)} \\
&\quad + \frac{n_1\left(1-\theta\exp\left(1-e^{\theta}\right)\right)}{\pi_0 + \theta\exp\left(1-e^{\theta}\right)\left(1-\pi_0-\pi_1\right)} - \left(\frac{n-n_0+n_1}{1-\pi_0-\pi_1}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial LL}{\partial \pi_1} &= -\frac{n_0\exp\left(1-e^{\theta}\right)}{\pi_0 + \exp\left(1-e^{\theta}\right)\left(1-\pi_0-\pi_1\right)} \\
&\quad - \frac{n_1\exp\left(1-e^{\theta}\right)}{\pi_0 + \theta\exp\left(1-e^{\theta}\right)\left(1-\pi_0-\pi_1\right)} - \left(\frac{n-n_0-n_1}{1-\pi_0-\pi_1}\right).
\end{aligned}
$$

The Newton-Raphson method, a numerical process, was employed in this work to determine the maximum likelihood estimates of $\hat{\theta}, \hat{\pi}_0$ and $\hat{\pi}_1$ using the 'countDM' package in the R program [11].

5 Application to Real Dataset

In this section, two different sets of data will be used to show the performances of the zero-one inflated Bell distribution by comparing it to the zero inflated Poisson (ZIP), zero-one inflated Poisson (ZOIP), and zero inflated Bell (ZIB) distributions. The maximum likelihood method provides parameter estimations. The minus log-likelihood (-log L), Akaike information criterion (AIC), and Bayesian information criterion (BIC) are used for model selection and evaluation [12].

5.1 Dataset 1: the claim counts from the motor insurance

The dataset is the number of claims from a motor insurance. The data includes 2,812 complete records from policyholders who remained with the company over the past year. The dataset was sourced from the SAS Enterprise Miner database [13]. The frequency data is presented in Table 1. Table 2 presents the estimated parameters obtained through the maximum likelihood method for the ZIP, ZOIP, ZIB, and ZOIB distributions pertaining to dataset 1. The results, including minus log-likelihood, AIC, and BIC, indicate that the ZOIB distribution outperforms the other three models.

Table 1: The number of claims and frequency from a motor insurance.

number of claims	0	1	2	3	4	5
frequency	1706	351	408	268	74	5

Table 2: The maximum likelihood estimator of distributions and some criteria of model performance for dataset 1.

	ZIP	ZOIP	ZIB	ZOIB
	$\hat{\lambda} = 1.6899$ $\hat{\pi}_0 = 0.5177$	$\hat{\lambda} = 2.5708$ $\hat{\pi}_0 = 0.6067$ $\hat{\pi}_1 = 0.1248$	$\hat{\theta} = 0.6458$ $\hat{\pi}_0 = 0.3429$	$\hat{\theta} = 0.4785$ $\hat{\pi}_0 = 0.6102$ $\hat{\pi}_1 = 0.1251$
-log L	3347.59	3706.06	3406.86	**3340.97**
AIC	6699.19	7418.12	6817.72	**6687.94**
BIC	6711.06	7435.95	6829.60	**6705.76**

5.2 Dataset 2: automobile insurance Zaire 1974

The data show the number of automobile third-party insurance liability portfolios in Zaire in 1974. Denuit [14] used them, and the DDPM package in the R programming language now makes them available [15]. Table 3 provides the frequency data. Again, ZIP, ZOIP, ZIB, and ZOIB distributions have been fitted to data by the maximum likelihood method. Based on the results, there is sufficient statistical evidence that the ZOIB distribution fits the data quite well (Table 4).

Table 3: The number of automobile insurance third-party liability portfolios.

number of portfolios	0	1	2	3	4	5
frequency	3719	232	38	7	3	1

Table 4: The MLE estimator of distributions and some criteria of model performance for dataset 2.

	ZIP	ZOIP	ZIB	ZOIB
	$\hat{\lambda} = 0.4317$ $\hat{\pi}_0 = 0.7996$	$\hat{\lambda} = 0.8522$ $\hat{\pi}_0 = 0.9049$ $\hat{\pi}_1 = 0.0368$	$\hat{\theta} = 0.2010$ $\hat{\pi}_0 = 0.6480$	$\hat{\theta} = 0.3129$ $\hat{\pi}_0 = 0.9298$ $\hat{\pi}_1 = 0.0580$
-log L	1187.78	1183.92	1185.53	**1183.58**
AIC	2379.56	2373.83	2375.05	**2373.15**
BIC	2392.15	2392.72	**2387.65**	2392.04

6 Conclusion

In this paper, we introduced the zero-one inflated Bell distribution which adds heterogeneity to the Bell distribution for count data by adding extra zeros and ones. We demonstrated some statistical properties of the ZOIB distribution, including the probability mass function, probability generating function, moment, mean, and variance. We also presented the sub-models for the ZOIB distribution. We used the maximum likelihood method to estimate the parameters of the ZOIB distribution. This distribution was implemented on two real datasets from the insurance field, which illustrated the usefulness and compared to the related distributions, such as the zero inflated Poisson, zero-one inflated Poisson and zero inflated Bell distributions. The results for the fitting performance of various distributions, assessed through the minus log-likelihood, Akaike information criterion and Bayesian information criterion, indicate that the ZOIB distribution demonstrates notable flexibility.

We are hopeful that the proposed distribution may attract wider applications in analyzing count data that have an excess of zeros and ones.

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