International Journal of Mathematics and Computer Science
Volume **20,** Issue no. 2, (2025), 671–678
DOI: https://doi.org/10.69793/ijmcs/02.2025/shukla

A Unified Characterization of Congruence Co-Kernels in Paradistributive Latticoids

Ravikumar Bandaru[1], **Prashant Patel**[1],
Ramesh Prasad Panda[1], **Rahul Shukla**[2]

[1]Department of Mathematics
VIT-AP University
Amaravati, Andhra Pradesh-522237, India

[2]Department of Mathematical Sciences and Computing
Walter Sisulu University
Mthatha 5117, South Africa

email: ravimaths83@gmail.com, prashant.patel9999@gmail.com and rameshprpanda@gmail.com, rshukla@wsu.ac.za

(Received February 1, 2025, Revised February 24, 2025,
Accepted February 24, 2025, Published April 7, 2025)

Abstract

In this paper, we characterize congruence co-kernels in a parapseudo-complemented paradistributive latticoid.

1 Introduction

Bandaru et al. [4] generalized the concept of a distributive lattice by introducing the notion of a Paradistributive Latticoid (PDL) and investigated its fundamental properties. They also established a subdirect representation of a PDL. More recently, Bandaru et al. [2] introduced the concept of parapseudo-complementation in a PDL and explored its elementary properties. Additionally, they derived necessary conditions for a PDL with a minimal element to be parapseudo-complemented and examined the conditions

Key words and phrases: Paradistributive latticoid (PDL), parapseudo-complementation, filter, co-kernel.
AMS (MOS) Subject Classifications: 06A12, 06D15, 06D99.
The corresponding author is R. Shukla
ISSN 1814-0432, 2025, https://future-in-tech.net

under which parapseudo-complementation can be equationally defined. Furthermore, they established a one-to-one correspondence between the set of all minimal elements and the set of all parapseudo-complementations. Later, Bandaru et al. [5] introduce the concept of a normal paradistributive latticoid and characterized in terms of the prime filters and minimal prime filters. Also, Ajjarapu et al. [3] studied the concepts of prime filters and minimal prime filters on a paradistributive latticoid (PDL) and discussed various results. In addition, they proved that the annihilator filter of a non-empty subset of a PDL is equal to the intersection of all prime flters not containing the subset. Furthermore, they explored minimal prime filters and studied certain results associated with them. Also, they derived some topological properties of the space of prime filters and minimal prime filters.

In this paper, we characterize the congruence co-kernels in a parapseudo-complemented PDL.

2 Preliminaries

First, we recall the necessary definitions and results from [4, 2].

Definition 2.1. *[4] An algebra $(V, \vee, \wedge, 1)$ of type (2,2,0) is called a Paradistributive Latticoid, abbreviated as PDL, if it assures the subsequent axioms:*
$(LD\vee)$ $\wp_1 \vee (\wp_2 \wedge \wp_3) = (\wp_1 \vee \wp_2) \wedge (\wp_1 \vee \wp_3)$.
$(RD\vee)$ $(\wp_1 \wedge \wp_2) \vee \wp_3 = (\wp_1 \vee \wp_3) \wedge (\wp_2 \vee \wp_3)$.
(L_1) $(\wp_1 \vee \wp_2) \wedge \wp_2 = \wp_2$.
(L_2) $(\wp_1 \vee \wp_2) \wedge \wp_1 = \wp_1$.
(L_3) $\wp_1 \vee (\wp_1 \wedge \wp_2) = \wp_1$.
(I_1) $\wp_1 \vee 1 = 1$.
for any $\wp_1, \wp_2, \wp_3 \in V$.

For any $\wp_1, \wp_2 \in V$, we say that $\wp_1$ is less than or equal to $\wp_2$ and write $\wp_1 \leq \wp_2$ if $\wp_1 \wedge \wp_2 = \wp_1$ or equivalently $\wp_1 \vee \wp_2 = \wp_2$ and it can be easily observed that $\leq$ is a partial order on V. The element 1, in Definition 2.1, is called the greatest element.

Lemma 2.2. *[4] Let $(V, \vee, \wedge, 1)$ be a PDL. Then for any $\wp_1, \wp_2, \wp_3, \wp_4 \in V$, we have the following:*
(1) $1_L \wedge \wp_1 = \wp_1$.
(2) $\wp_1 \wedge 1_L = \wp_1$.
(3) $1_L \vee \wp_1 = 1_L$.
(4) $(\wp_1 \vee \wp_2) \wedge \wp_3 = (\wp_1 \wedge \wp_3) \vee (\wp_2 \wedge \wp_3)$.

(5) $\wp_1 \vee (\wp_2 \wedge \wp_3) = \wp_1 \vee (\wp_3 \wedge \wp_2)$.
(6) *The operation* $\vee$ *is associative in* V *i.e.,* $\wp_1 \vee (\wp_2 \vee \wp_3) = (\wp_1 \vee \wp_2) \vee \wp_3$.
(7) *The set* $V_{\intercal_1} = \{\wp_1 \in V \mid \intercal_1 \leq \wp_1\} = \{\intercal_1 \vee \wp_1 \mid \wp_1 \in V\}$ *is a distributive lattice under induced operations* $\vee$ *and* $\wedge$ *with* $\intercal_1$ *as its least element.*
(8) $\wp_4 \vee \{\wp_1 \wedge (\wp_2 \wedge \wp_3)\} = \wp_4 \vee \{(\wp_1 \wedge \wp_2) \wedge \wp_3\}$.
(9) $\wp_1 \vee (\wp_2 \vee \wp_3) = \wp_1 \vee (\wp_3 \vee \wp_2)$.
(10) $\wp_1 \vee \wp_2 = 1$ *if and only if* $\wp_2 \vee \wp_1 = 1$.
(11) $\wp_1 \wedge \wp_2 = \wp_2 \wedge \wp_1$ *whenever* $\wp_1 \vee \wp_2 = 1$.

Let V be a PDL. Then, an element $\intercal_1 \in V$ is said to be a minimal element if for any $u \in V$, $u \leq \intercal_1 \Rightarrow u = \intercal_1$.

Lemma 2.3. *[4] Let V be a PDL. Then, for any $\intercal_1 \in V$, the following are equivalent:*
(1). $\intercal_1$ *is minimal*
(2). $\wp_1 \wedge \intercal_1 = \intercal_1$ *for all* $\wp_1 \in V$
(3). $\wp_1 \vee \intercal_1 = \wp_1$ *for all* $\wp_1 \in V$.

Definition 2.4. *[4] A non-empty subset F of a PDL V is said to be a filter if it satisfies the following:*

$$\wp_1, \wp_2 \in F \Rightarrow \wp_1 \wedge \wp_2 \in F.$$
$$\wp_1 \in F, \intercal_1 \in V \Rightarrow \intercal_1 \vee \wp_1 \in F.$$

Theorem 2.5. *[4] Let S be a non-empty subset of V. Then*

$$[S) \;\; = \;\; \{\wp_1 \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, \wp_1 \in V, 1_L \leq i \leq n \text{ and } n \text{ is a positive integer }\}$$

is the smallest filter of V containing S.

Lemma 2.6. *[4] Let V be a VDL and F be a filter of V. Then for any $\wp_1, \wp_2 \in V$, we have the following:*
(1) $[\wp_1) = \{\hbar_1 \vee \wp_1 \mid \hbar_1 \in V\}$.
(2) $\wp_1 \in [\wp_2)$ *if and only if* $\wp_1 = \wp_1 \vee \wp_2$ *for all* $\wp_1, \wp_2 \in V$.
(3) $\wp_1 \vee \wp_2 \in F$ *if and only if* $\wp_2 \vee \wp_1 \in F$.
(4) $[\wp_1 \vee \wp_2) = [\wp_2 \vee \wp_1)$.
(5) $[\wp_1 \wedge \wp_2) = [\wp_2 \wedge \wp_1) = [\wp_1) \vee [\wp_2)$.

Definition 2.7. *[2] Let $(V, \vee, \wedge, 1)$ be a Paradistributive Latticoid (PDL) and consider a unary operation denoted as $\hbar_1 \mapsto \hbar_1^{\blacklozenge}$ on V. This operation is called a parapseudo-complementation on V if it satisfies the following conditions:*

(1) *If* $\hbar_1 \vee \hbar_2 = 1$, *then* $\hbar_1 \vee {\hbar_2}^{\blacklozenge} = \hbar_1$.
(2) $\hbar_1 \vee {\hbar_1}^{\blacklozenge} = 1$.
(3) $(\hbar_1 \wedge \hbar_2)^{\blacklozenge} = {\hbar_1}^{\blacklozenge} \vee {\hbar_2}^{\blacklozenge}$.

If there is no ambiguity about the parapseudo-complementation on a PDL V, we can say that V is a parapseudo-complemented PDL (PPDL).

3 Characterization of Congruence Co-Kernels

In this section, we give different sets of equations which characterize the parapseudo-complementation on a PDL. That is, a parapseudo-complementation on a PDL is equationally definable. We also define the co-kernel of a congruence on a PDL and characterize the congruence co-kernels in a parapseudo-complemented PDL. First, we begin with the following.

Lemma 3.1. *Let V be a PDL and $\blacklozenge$ a parapseudo-complementation on V. Then, for any $a, b \in V$, the following hold:*
(*i*) $(\mathsf{T}_1 \vee \mathsf{T}_2)^{\blacklozenge} = (\mathsf{T}_2 \vee \mathsf{T}_1)^{\blacklozenge}$
(*ii*) $(\mathsf{T}_1 \wedge \mathsf{T}_2)^{\blacklozenge} = (\mathsf{T}_2 \wedge \mathsf{T}_1)^{\blacklozenge}$

Proof. Let $\mathsf{T}_1, \mathsf{T}_2 \in V$. Then
(*i*) $(\mathsf{T}_1 \vee \mathsf{T}_2)^{\blacklozenge} = (\mathsf{T}_1 \vee \mathsf{T}_2)^{\blacklozenge\blacklozenge\blacklozenge} = (\mathsf{T}_1^{\blacklozenge\blacklozenge} \vee \mathsf{T}_2^{\blacklozenge\blacklozenge})^{\blacklozenge} = (\mathsf{T}_2^{\blacklozenge\blacklozenge} \vee \mathsf{T}_1^{\blacklozenge\blacklozenge})^{\blacklozenge} = (\mathsf{T}_2 \vee \mathsf{T}_1)^{\blacklozenge\blacklozenge\blacklozenge} = (\mathsf{T}_2 \vee \mathsf{T}_1)^{\blacklozenge}$.
(*ii*) $(\mathsf{T}_1 \wedge \mathsf{T}_2)^{\blacklozenge} = \mathsf{T}_1^{\blacklozenge} \vee \mathsf{T}_2^{\blacklozenge} = \mathsf{T}_2^{\blacklozenge} \vee \mathsf{T}_1^{\blacklozenge} = (\mathsf{T}_2 \wedge \mathsf{T}_1)^{\blacklozenge}$. □

The following theorem shows that the parapseudo-complementation on a PDL is equationally definable.

Theorem 3.2. *[2] A unary operation $\blacklozenge$ on a PDL V is a parapseudo-complementation on V if and only if it satisfies the following equations:*
(1) $\mathsf{T}_1 \vee \mathsf{T}_1^{\blacklozenge} = 1$
(2) $\mathsf{T}_1 \wedge \mathsf{T}_1^{\blacklozenge\blacklozenge} = \mathsf{T}_1^{\blacklozenge\blacklozenge}$
(3) $(\mathsf{T}_1 \wedge \mathsf{T}_2)^{\blacklozenge} = \mathsf{T}_1^{\blacklozenge} \vee \mathsf{T}_2^{\blacklozenge}$
(4) $(\mathsf{T}_1 \vee \mathsf{T}_2)^{\blacklozenge\blacklozenge} = \mathsf{T}_1^{\blacklozenge\blacklozenge} \vee \mathsf{T}_2^{\blacklozenge\blacklozenge}$
(5) $\mathsf{T}_1 \vee 1^{\blacklozenge} = \mathsf{T}_1$

In the following theorem, we give a different set of equations that characterize a parapseudo-complementation on a PDL.

Theorem 3.3. *[2] A unary operation* $\blacklozenge$ *on a PDL* V *is a parapseudo-complementation on* V *if and only if it satisfies the following conditions:*

(1) $\mathsf{T}_2 \vee \mathsf{T}_1^{\blacklozenge} = \mathsf{T}_2 \vee (\mathsf{T}_1 \vee \mathsf{T}_2)^{\blacklozenge}$

(2) $\mathsf{T}_1 \vee 1^{\blacklozenge} = \mathsf{T}_1$

(3) $1^{\blacklozenge\blacklozenge} = 1$

(4) $(\mathsf{T}_1 \wedge \mathsf{T}_2)^{\blacklozenge} = \mathsf{T}_1^{\blacklozenge} \vee \mathsf{T}_2^{\blacklozenge}$

Definition 3.4. *[1] A variety* ν *is termed subtractive if there exists a binary operation* $\varsigma(\hbar_1, \hbar_2)$ *within* ν *satisfying the identities* $\varsigma(\hbar_1, \hbar_1) = 1$ *and* $\varsigma(\hbar_1, 1) = \hbar_1$

Since parapseudo-complementation in a PDL is definable through equations, the class of parapseudo-complemented PDLs constitutes a variety. This leads to the following assertion.

Lemma 3.5. *The variety of parapseudo-complemented PDLs is subtractive.*

Proof. Let ν denote the variety of pseudo-complemented PDLs. Define the binary term $\varsigma(\hbar_1, \hbar_2) = \hbar_1 \vee \hbar_2^{\blacklozenge}$. Then, it follows that: $\varsigma(\hbar_1, \hbar_1) = 1, \varsigma(\hbar_1, 1) = \hbar_1$. Thus, ν satisfies the conditions for being subtractive. $\square$

We use $\mathrm{Con}(\mathcal{A})$ to represent the collection of all congruences on an algebra $\mathcal{A}$.

Definition 3.6. *[6] A PDL* $(V, \vee, \wedge, 1)$ *equipped with a parapseudo-complementation* $\blacklozenge$ *is associated with a binary relation* $\mathcal{R}$ *on* V. *When* $\mathcal{R}$ *is a congruence relation, the set:* $[1]_{\mathcal{R}} = \{\hbar_1 \in V \mid (\hbar_1, 1) \in \mathcal{R}\}$ *is known as a congruence co-kernel.*

Definition 3.7. *An algebra* $\mathcal{A} = (A, F)$ *with a nullary operation* 1 *is said to be *permutable at** 1 *if for any* $\theta, \phi \in Con(\mathcal{A})$*, the equality holds:* $[1]_{\theta\circ\phi} = [1]_{\phi\circ\theta}$. *A variety* ν *is permutable at* 1 *if every algebra in* ν *exhibits this property.*

Lemma 3.8. *[6] A variety* ν *with* 1 *is permutable at* 1 *if and only if it is subtractive.*

Lemma 3.9. *[6] Let* ν *be a variety with* 1. *For any algebra* $\mathcal{A} = (A, F)$ *in* ν *and any binary relation* $\mathcal{R}$ *on* A*, let* $\theta(\mathcal{R})$ *denote the smallest congruence on* $\mathcal{A}$ *containing* $\mathcal{R}$. *The following statements are equivalent:*

(a) ν *is permutable at* 1.

(b) For every algebra $\mathcal{A} \in \nu$ *and for every reflexive and compatible relation* $\mathcal{R}$ *on* $\mathcal{A}$*, it holds that:* $[1]_{\mathcal{R}} = [1]_{\theta(\mathcal{R})}$*, where* $[1]_{\mathcal{R}} = \{\hbar_1 \in A : (\hbar_1, 1) \in \mathcal{R}\}$.

Proof. $(a) \Rightarrow (b)$: Suppose ν is permutable at 1 and let $\varsigma(\hbar_1, \hbar_2)$ be a binary operation in ν such that $\varsigma(\hbar_1, \hbar_1) = 1$ and $\varsigma(\hbar_1, 1) = \hbar_1$. Given an algebra $\mathcal{A} = (A, F) \in \nu$ and a reflexive, compatible relation $\mathcal{R}$ on $\mathcal{A}$, we observe that $\mathcal{R}^{-1}$ and $\mathcal{R} \circ \mathcal{R}$ are also reflexive and compatible. Then:

(i) If $\hbar_1 \in [1]_{\mathcal{R}^{-1}}$, then $(\hbar_1, 1) \in \mathcal{R}^{-1}$, implying that $(1, \hbar_1) = (\varsigma(\hbar_1, \hbar_1), \varsigma(\hbar_1, 1)) \in \mathcal{R}^{-1}$. Thus, $(\hbar_1, 1) \in \mathcal{R}$, meaning $\hbar_1 \in [1]_{\mathcal{R}}$, and so $[1]_{\mathcal{R}^{-1}} \subseteq [1]_{\mathcal{R}}$.

(ii) If $\hbar_2 \in [1]_{\mathcal{R}\circ\mathcal{R}}$, there exists $\hbar_1 \in A$ such that $(\hbar_2, \hbar_1) \in \mathcal{R}$ and $(\hbar_1, 1) \in \mathcal{R}$. Consequently, $(\hbar_2, 1) = (\varsigma(\hbar_2, \varsigma(\hbar_1, \hbar_1)), \varsigma(\hbar_1, \varsigma(\hbar_1, 1))) \in \mathcal{R}$, which implies $\hbar_2 \in [1]_{\mathcal{R}}$, proving that $[1]_{\mathcal{R}\circ\mathcal{R}} \subseteq [1]_{\mathcal{R}}$.

By induction using (i) and (ii), we establish that $[1]_{\theta(\mathcal{R})} \subseteq [1]_{\mathcal{R}}$, and since the reverse inclusion is immediate, (b) follows.

$(b) \Rightarrow (a)$: Consider $F_\nu(\hbar_1)$, the free algebra in ν with one free generator $\hbar_1$, and define the reflexive, compatible relation R generated by $(\hbar_1, 1)$. By assumption, since $(1, \hbar_1) \in \theta(\mathcal{R})$, it follows that $(1, \hbar_1) \in \mathcal{R}$, ensuring the existence of a binary term $\varsigma(\hbar_1, \hbar_2)$ such that $1 = \varsigma(\hbar_1, \hbar_1)$ and $\hbar_1 = \varsigma(\hbar_1, 1)$. By the Lemma 3.8, ν is thus permutable at 1. □ □

We now proceed to establish the following theorem.

Lemma 3.10. *Let $(V, \vee, \wedge, 0)$ be a PDL. Then for any $\top_1, \top_2, \top_3 \in V, \top_1 \vee \top_3 = 1$ and $\top_2 \vee \top_3 = 1 \Rightarrow \top_3 \vee (\top_1 \wedge \top_2) = 1$.*

Proof. Let $\top_1, \top_2, \top_3 \in V, \top_1 \vee \top_3 = 1$ and $\top_2 \vee \top_3 = 1$. Then $\top_3 \vee \top_1 = 1$ and $\top_3 \vee \top_2 = 1$ and hence $\top_3 \vee (\top_1 \wedge \top_2) = 1$. □

The following theorem characterizes the congruence co-kernels in parapseudo-complemented PDLs.

Theorem 3.11. *Let $(V, \vee, \wedge, 1)$ be a PDL and $\blacklozenge$ a parapseudo-complementation on V and $\emptyset \neq F \subseteq L$. Then F is a co-kernel of some congruence on V if and only if F satisfies the following conditions:*

(i) *F is a filter of V*

(ii) *$\hbar_1 \in F \Rightarrow \hbar_1^{\blacklozenge\blacklozenge} \in F$*

(iii) *if $\hbar_1, \hbar_2 \in F$ then $(\hbar_1 \wedge \hbar_2)^{\blacklozenge\blacklozenge} \in F$.*

Proof. Suppose $F = [1]_\theta$ for some congruence θ on L. Then it can be directly verified that F is a filter of L. Now $\hbar_1 \in F$ implies that $(\hbar_1, 1) \in \theta$. Hence $(\hbar_1^{\blacklozenge}, 1^{\blacklozenge}) \in \theta$ so that $(\hbar_1^{\blacklozenge\blacklozenge}, 1^{\blacklozenge\blacklozenge}) \in \theta$. Therefore $(\hbar_1^{\blacklozenge\blacklozenge}, 1) \in \theta$ since $1^{\blacklozenge\blacklozenge} = 1$. Thus $\hbar_1^{\blacklozenge\blacklozenge} \in F$. Now, suppose $\hbar_1, \hbar_2 \in F$. Then $(\hbar_1, 1), (\hbar_2, 1) \in \theta$ and hence $(\hbar_1^{\blacklozenge}, 1^{\blacklozenge}), (\hbar_2^{\blacklozenge}, 1^{\blacklozenge}) \in \theta$. Therefore $((\hbar_1^{\blacklozenge} \vee \hbar_2^{\blacklozenge})^{\blacklozenge}, 1) = ((\hbar_1^{\blacklozenge} \vee \hbar_2^{\blacklozenge})^{\blacklozenge}, (1^{\blacklozenge} \vee 1^{\blacklozenge})^{\blacklozenge}) \in \theta$. Thus $(\hbar_1^{\blacklozenge} \vee \hbar_2^{\blacklozenge})^{\blacklozenge} \in F$ and hence, by Definition 2.7(3), $(\hbar_1 \wedge \hbar_2)^{\blacklozenge\blacklozenge} \in$

F. Conversely, assume that $\emptyset \neq F \subseteq L$ satisfies $(i), (ii)$ and (iii). Define a binary relation R on L by

$$(\hbar_1, \hbar_2) \in R \text{ if and only if } \hbar_1 \vee \hbar_2{}^{\blacklozenge} \in F \text{ and } \hbar_2 \vee \hbar_1{}^{\blacklozenge} \in F.$$

Clearly, R is reflexive. Now, we prove the compatibility of R.
Let $(\mathsf{T}_1, \mathsf{T}_2) \in R$. Then $\mathsf{T}_1 \vee \mathsf{T}_2^{\blacklozenge} \in F$ and $\mathsf{T}_2 \vee \mathsf{T}_1^{\blacklozenge} \in F$. Hence, by (ii), $(\mathsf{T}_1 \vee \mathsf{T}_2^{\blacklozenge})^{\blacklozenge\blacklozenge} \in F$ and $(\mathsf{T}_2 \vee \mathsf{T}_1^{\blacklozenge})^{\blacklozenge\blacklozenge} \in F$. Hence $\mathsf{T}_1^{\blacklozenge} \vee \mathsf{T}_2^{\blacklozenge\blacklozenge} \in F$ and $\mathsf{T}_2^{\blacklozenge} \vee \mathsf{T}_1^{\blacklozenge\blacklozenge} \in F$. Therefore $(\mathsf{T}_1^{\blacklozenge}, \mathsf{T}_2^{\blacklozenge}) \in R$. Now, let $(\mathsf{T}_1, \mathsf{T}_2) \in R$ and $(\mathsf{T}_3, \mathsf{T}_4) \in R$. Then $\mathsf{T}_1 \vee \mathsf{T}_2^{\blacklozenge} \in F, \mathsf{T}_2 \vee \mathsf{T}_1^{\blacklozenge} \in F, \mathsf{T}_3 \vee \mathsf{T}_4^{\blacklozenge} \in F$ and $\mathsf{T}_4 \vee \mathsf{T}_3^{\blacklozenge} \in F$. Therefore, $\mathsf{T}_1 \vee \mathsf{T}_2^{\blacklozenge} \vee \mathsf{T}_4^{\blacklozenge} \in F$ since $\mathsf{T}_4^{\blacklozenge} \vee (\mathsf{T}_1 \vee \mathsf{T}_2^{\blacklozenge}) \in F$ by Definition 2.4 and Lemma 2.6(3). Also, $\mathsf{T}_3 \vee \mathsf{T}_4^{\blacklozenge} \vee \mathsf{T}_2^{\blacklozenge} \in F$ since $\mathsf{T}_2^{\blacklozenge} \vee (\mathsf{T}_3 \vee \mathsf{T}_4^{\blacklozenge}) \in F$ by Definition 2.4 and Lemma 2.6(3). Hence, by Definition 2.7(3) and Definition 2.1(RD $\vee$),

$$(\mathsf{T}_1 \wedge \mathsf{T}_3) \vee (\mathsf{T}_2 \wedge \mathsf{T}_4)^{\blacklozenge} = (\mathsf{T}_1 \wedge \mathsf{T}_3) \vee (\mathsf{T}_2^{\blacklozenge} \vee \mathsf{T}_4^{\blacklozenge}) = (\mathsf{T}_1 \vee \mathsf{T}_2^{\blacklozenge} \vee \mathsf{T}_4^{\blacklozenge}) \wedge (\mathsf{T}_3 \vee \mathsf{T}_2^{\blacklozenge} \vee \mathsf{T}_4^{\blacklozenge}) \in F.$$

Similarly, we can prove that $(\mathsf{T}_2 \wedge \mathsf{T}_4) \vee (\mathsf{T}_1 \wedge \mathsf{T}_3)^{\blacklozenge} \in F$. Thus $(\mathsf{T}_1 \wedge \mathsf{T}_3, \mathsf{T}_2 \wedge \mathsf{T}_4) \in R$. Since $\mathsf{T}_1 \vee \mathsf{T}_2^{\blacklozenge} \in F$ and $\mathsf{T}_3 \vee \mathsf{T}_4^{\blacklozenge} \in F$, we have, by(iii) and Definition 2.7(3),

$$((\mathsf{T}_1 \vee \mathsf{T}_2^{\blacklozenge})^{\blacklozenge} \vee (\mathsf{T}_3 \vee \mathsf{T}_4^{\blacklozenge})^{\blacklozenge})^{\blacklozenge} \in F \text{ and } ((\mathsf{T}_2 \vee \mathsf{T}_1^{\blacklozenge})^{\blacklozenge} \vee (\mathsf{T}_4 \vee \mathsf{T}_3^{\blacklozenge})^{\blacklozenge})^{\blacklozenge} \in F \qquad (A)$$

It can be easily verified that
$(\mathsf{T}_4^{\blacklozenge} \vee \mathsf{T}_3)^{\blacklozenge} \vee (\mathsf{T}_2^{\blacklozenge} \vee \mathsf{T}_1)^{\blacklozenge} \vee \mathsf{T}_1 \vee \mathsf{T}_3 \vee \mathsf{T}_2^{\blacklozenge} = 1$ and $(\mathsf{T}_4^{\blacklozenge} \vee \mathsf{T}_3)^{\blacklozenge} \vee (\mathsf{T}_2^{\blacklozenge} \vee \mathsf{T}_1)^{\blacklozenge} \vee \mathsf{T}_1 \vee \mathsf{T}_3 \vee \mathsf{T}_4^{\blacklozenge} = 1$.
Hence, by Lemma 3.10, we get

$$(\mathsf{T}_4^{\blacklozenge} \vee \mathsf{T}_3)^{\blacklozenge} \vee (\mathsf{T}_2^{\blacklozenge} \vee \mathsf{T}_1)^{\blacklozenge} \vee \mathsf{T}_1 \vee \mathsf{T}_3 \vee (\mathsf{T}_2^{\blacklozenge} \wedge \mathsf{T}_4^{\blacklozenge}) = 1$$

Now, by Definition 2.7(1), we get
$(\mathsf{T}_4^{\blacklozenge} \vee \mathsf{T}_3)^{\blacklozenge} \vee (\mathsf{T}_2^{\blacklozenge} \vee \mathsf{T}_1)^{\blacklozenge} \vee \mathsf{T}_1 \vee \mathsf{T}_3 \vee (\mathsf{T}_2^{\blacklozenge} \wedge \mathsf{T}_4^{\blacklozenge})^{\blacklozenge} = (\mathsf{T}_4^{\blacklozenge} \vee \mathsf{T}_3)^{\blacklozenge} \vee (\mathsf{T}_2^{\blacklozenge} \vee \mathsf{T}_1)^{\blacklozenge} \vee \mathsf{T}_1 \vee \mathsf{T}_3$.
Hence, by Definition 2.7(3), we get
$(\mathsf{T}_4^{\blacklozenge} \vee \mathsf{T}_3)^{\blacklozenge} \vee (\mathsf{T}_2^{\blacklozenge} \vee \mathsf{T}_1)^{\blacklozenge} \vee \mathsf{T}_1 \vee \mathsf{T}_3 \vee (\mathsf{T}_2^{\blacklozenge\blacklozenge} \vee \mathsf{T}_4^{\blacklozenge\blacklozenge}) = (\mathsf{T}_4^{\blacklozenge} \vee \mathsf{T}_3)^{\blacklozenge} \vee (\mathsf{T}_2^{\blacklozenge} \vee \mathsf{T}_1)^{\blacklozenge} \vee \mathsf{T}_1 \vee \mathsf{T}_3$.
Therefore, by Theorem 3.2(4), we get
$(\mathsf{T}_4^{\blacklozenge} \vee \mathsf{T}_3)^{\blacklozenge} \vee (\mathsf{T}_2^{\blacklozenge} \vee \mathsf{T}_1)^{\blacklozenge} \vee \mathsf{T}_1 \vee \mathsf{T}_3 \vee (\mathsf{T}_2 \vee \mathsf{T}_4)^{\blacklozenge\blacklozenge} = (\mathsf{T}_4^{\blacklozenge} \vee \mathsf{T}_3)^{\blacklozenge} \vee (\mathsf{T}_2^{\blacklozenge} \vee \mathsf{T}_1)^{\blacklozenge} \vee \mathsf{T}_1 \vee \mathsf{T}_3$.
Since $\mathsf{T}_1 \vee \mathsf{T}_2 = 1$ when $\mathsf{T}_1 \vee \mathsf{T}_2^{\blacklozenge} = \mathsf{T}_1$, we have

$$(\mathsf{T}_4^{\blacklozenge} \vee \mathsf{T}_3)^{\blacklozenge} \vee (\mathsf{T}_2^{\blacklozenge} \vee \mathsf{T}_1)^{\blacklozenge} \vee \mathsf{T}_1 \vee \mathsf{T}_3 \vee (\mathsf{T}_2 \vee \mathsf{T}_4)^{\blacklozenge} = 1.$$

Hence, by Lemma 3.1(10), $\mathsf{T}_1 \vee \mathsf{T}_3 \vee (\mathsf{T}_2 \vee \mathsf{T}_4)^{\blacklozenge} \vee (\mathsf{T}_4^{\blacklozenge} \vee \mathsf{T}_3)^{\blacklozenge} \vee (\mathsf{T}_2^{\blacklozenge} \vee \mathsf{T}_1)^{\blacklozenge} = 1$. Thus, by Definition 2.7(1),

$$\mathsf{T}_1 \vee \mathsf{T}_3 \vee (\mathsf{T}_2 \vee \mathsf{T}_4)^{\blacklozenge} \vee ((\mathsf{T}_4^{\blacklozenge} \vee \mathsf{T}_3)^{\blacklozenge} \vee (\mathsf{T}_2^{\blacklozenge} \vee \mathsf{T}_1)^{\blacklozenge})^{\blacklozenge} = \mathsf{T}_1 \vee \mathsf{T}_3 \vee (\mathsf{T}_2 \vee \mathsf{T}_4)^{\blacklozenge}.$$

Hence, by (A), we get $\mathsf{T}_1 \vee \mathsf{T}_3 \vee (\mathsf{T}_2 \vee \mathsf{T}_4)^{\blacklozenge} \in F$. Similarly, we can show that $(\mathsf{T}_2 \vee \mathsf{T}_4) \vee (\mathsf{T}_1 \vee \mathsf{T}_3)^{\blacklozenge} \wedge \in F$. Thus $(\mathsf{T}_1 \vee \mathsf{T}_3, \mathsf{T}_2 \vee \mathsf{T}_4) \in R$. Hence, R is reflexive and compatible relation on L. Also, if $\mathsf{T}_1 \in F$ then $\mathsf{T}_1 \vee 1^{\blacklozenge} = \mathsf{T}_1 \in F$ and $1 \vee \mathsf{T}_1^{\blacklozenge} = 1 \in F$. Therefore, $(\mathsf{T}_1, 1) \in R$, and hence $\mathsf{T}_1 \in [1]_R$. If $\mathsf{T}_1 \in [1]_R$ then $(\mathsf{T}_1, 1) \in R$ and hence $\mathsf{T}_1 = \mathsf{T}_1 \vee 1^{\blacklozenge} \in F$. Therefore $[1]_R = F$. Hence, by Lemma 3.9, $F = [1]_{\theta(R)}$, which is a congruence co-kernel. $\square$

References

[1] P. Agiliano, A. Ursini, On subtractive varieties III: From ideals to congruences, Algebra Universalis, **37,** (1997), 296–333.

[2] S. Ajjarapu, R. Bandaru, R. Shukla, Y. B. Jun, Parapseudo-complementation on paradistributive latticoids, European Journal of Pure and Applied Mathematics, **17,** no. 2, (2024), 1129–1145. https://doi.org/10.29020/nybg.ejpam.v17i2.5042.

[3] S. Ajjarapu, R. Bandaru, R. K. K, R. Shukla, Topological Properties of Prime Filters and Minimal PrimeFilters on a Paradistributive Latticoid, Internation Journal of Mathematics and Mathematical Sciences, 2024, Article ID 1862245, 12 pages.

[4] R. Bandaru, S. Ajjarapu, Paradistributive latticoids, European Journal of Pure and Applied Mathematics, **17,** no. 2, (2024), 819–834. https://doi.org/10.29020/nybg.ejpam.v17i2.5125.

[5] R. Bandaru, P. Patel, N. Rafi, R. Shukla, S. Ajjarapu, Normal Paradistributive Latticoids, European Journal of Pure and Applied Mathematics, **17,** no. 2, (2024), 1306–1320.

[6] I. Chajda, Varieties which are Locally Regular and Permutable at 0, Acta Univ. Palacki.Olomuc., Fac. rer. nat., Mathematica, **40,** (2001), 31–35.

[7] I. Chajda, A characterization of Congruence Kernels in Pseudocomplemented Semilattices, Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica, **41,** (2002), 35–37.