International Journal of Mathematics and Computer Science
Volume **20,** Issue no. 2, (2025), 501–506
DOI: https://doi.org/10.69793/ijmcs/02.2025/shinsho

On the Generalized Ramanujan-Nagell Equation $x^2 + (6R_k - 1)^m = (3R_k)^n$

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(Received December 18, 2024, Accepted January 20, 2025,
Published January 21, 2025)

Abstract

In this paper, we consider the generalized Ramanujan-Nagell equation $x^2 + (6R_k - 1)^m = (3R_k)^n$ involving Near-Repdigits and we show that, under some conditions, it has only the positive integer solution $(x, m, n) = (3R_k - 1, 1, 2)$. The proof is based on the Jacobi Symbol and Zsigmondy's theorem.

1 Introduction

In 1913, Ramanujan [5] conjectured that the equation $x^2 + 7 = 2^n$ has only the positive integer solutions $(x, n) = (1, 3), (3, 4), (5, 5), (11, 7), (181, 15)$.

In 1960, Nagell [4] solved Ramanujan's conjecture. Let b and c be fixed relatively prime positive integers greater than one. Then the generalized Ramanujan-Nagell equation $x^2 + b^m = c^n$ in positive integers x, m and n has been studied by a number of authors (cf. [2], [6], [7], [8], [9]).

In particular, a repdigit that contains only the digit 1 is called a *repunit* [1]. The repunit R_k is defined as $R_k = \frac{10^k - 1}{9} = 111 \cdots 11$. Moreover, if the last digits of Repdigits are different, then they are called Near-Repdigits.

Key words and phrases: Generalized Ramanujan-Nagell equation, Near-Repdigits, Zsigmondy's theorem, integer solution.
AMS (MOS) Subject Classifications: 11D61.
ISSN 1814-0432, 2025, https://future-in-tech.net

Here, we make the following conjecture about generalized Ramanujan–Nagell equation involving Near-Repdigits.

Conjecture 1. *Let k be a positive integer. Then the equation*

$$x^2 + (6R_k - 1)^m = (3R_k)^n \tag{1.1}$$

has only the positive integer solution $(x, m, n) = (3R_k - 1, 1, 2)$.

A simple program of the Computational Algebra System MAGMA [BoCa] verifies that this is true in the range $1 \leq k \leq 100$. Our main result is the following:

Theorem 1.1. *Suppose that at least one of the following conditions is satisfied:*
(i) $k = 1$,
(ii) $k = 2$,
(iii) $6R_k - 1$ *has two prime factors with* $\left(\frac{3R_k}{5}\right) = -1$.
Then Conjecture 1 is true.

2 Preliminaries.

Lemma 2.1. (Jacobi Symbol) *Let p be an odd positive integer and let $a, b \in \mathbb{Z}$ be coprime to p. Then we have:*

(i) $\left(\frac{2}{p}\right) = \begin{cases} 1 \text{ if } p \equiv 1, 7 \pmod 8 \\ -1 \text{ if } p \equiv 3, 5 \pmod 8), \end{cases}$

(ii) $\left(\frac{-1}{p}\right) = \begin{cases} 1 \text{ if } p \equiv 1 \pmod 4 \\ -1 \text{ if } p \equiv 3 \pmod 4), \end{cases}$

(iii) $\left(\frac{a}{p}\right) = \begin{cases} 1 \cdot \left(\frac{p}{a}\right) \text{ if } a \equiv 1 \pmod 4 \text{ or } p \equiv 1 \pmod 4 \\ -1 \cdot \left(\frac{p}{a}\right) \text{ if } a \equiv 3 \pmod 4 \text{ and } p \equiv 3 \pmod 4). \end{cases}$

The following is a direct consequence of an old version of the Primitive Divisor Theorem due to Zsigmondy[10].

Lemma 2.2. (Zsigmondy [10]) *Let A and B relatively prime integers with $A > B \geq 1$. Let $\{a_k\}_{k \geq 1}$ be the sequence defined as*

$$a_k = A^k + B^k.$$

If $k > 1$, then a_k has a prime factor not dividing $a_1 a_2 \cdots a_{k-1}$, whenever $(A, B, k) \neq (2, 1, 3)$.

3 Main results

We first show that when $k = 1$, equation has only the positive integer solution $(x, m, n) = (2, 1, 2)$. The following Proposition easily follows from Theorem of Terai [7].

Proposition 3.1. (Terai [7]) *Let c be a positive integer with $c \geq 2$. If $c \leq 30$, then the equation*

$$x^2 + (2c-1)^m = c^n$$

has only the positive integer solution $(x, m, n) = (c-1, 1, 2)$ except for the cases $c = 12, 24$.

Our assertion is now immediate from Proposition 3.1. This completes the proof of Theorem 1.1 (i).

We next show that when $k = 2$, equation has only the positive integer solution $(x, m, n) = (32, 1, 2)$.

Proposition 3.2. (Muriefah, Luca and Togbé [3]) *Let $x \geq 1, y \geq 1, \gcd(x, y) = 1, n \geq 3, a \geq 0, b \geq 0$. Then the equation*

$$x^2 + 5^a 13^b = y^n$$

has no solution except for :
$n = 3$ $(x, y, a, b) = (70, 17, 0, 1), (142, 29, 2, 2)$;
$n = 4$ $(x, y, a, b) = (4, 3, 1, 1)$.

Lemma 3.3. *The equation $x^2 + 65^m = 33^n$ has only the positive integer solution $(x, m, n) = (32, 1, 2)$.*

We have $\left(\frac{33}{5}\right) = \left(\frac{3}{5}\right) = \left(\frac{5}{3}\right) = \left(\frac{2}{3}\right) = -1$, which $\left(\frac{*}{*}\right)$ is the Jacobi Symbol. Hence equation implies that n is even, say $n = 2$ from Proposition 3.2. From equation, we have $65^m = (33+x)(33-x)$. Since $\gcd(33+x, 33-x) = 2$, we obtain the follow two cases:

$$\begin{cases} 33 + x = (5 \cdot 13)^m \\ 33 - x = 1 \end{cases} \quad \text{or} \quad \begin{cases} 33 + x = 13^m \\ 33 - x = 5^m. \end{cases}$$

First, consider the case where adding these two equation yields $65^m + 1 = 66$, which has only the solution $m = 1$ by Lemma 2.2. Hence, the equation has only the solution $(x, m, n) = (32, 1, 2)$. Next, adding these two equation yields $5^m + 13^m = 66$. If m is even, then the equation modulo 3 implies that $2 \equiv 0 \pmod 3$, which is impossible. If m is odd, then the equation modulo

5 implies that $1 \equiv \{3, 2\} \pmod 5$, which is impossible. This completes the proof of Theorem 1.1 (ii).

Finally, we prove Theorem 1.1 (iii) using Jacobi Symbol and Zigmondy's theorem [10]. We have

$$\left(\frac{3R_k}{5}\right) = \left(\frac{3\cdots3}{5}\right) = \left(\frac{3}{5}\right) = \left(\frac{5}{3}\right) = \left(\frac{2}{3}\right) = -1,$$

which $\left(\frac{*}{*}\right)$ is the Jacobi Symbol. Hence equation implies that n is even, say $n = 2N$. From equation, we have $(3 \cdot 13 \cdots 3)^m = (3 \cdots 3^N + x)(3 \cdots 3^N - x)$. Since $\gcd(3 \cdots 3^N + x, 3 \cdots 3^N - x) = 2$, we obtain the follow two cases:

$$\begin{cases} 3 \cdots 3^N + x = (5 \cdot 13 \cdots 3)^m \\ 3 \cdots 3^N - x = 1 \end{cases} \quad \text{or} \quad \begin{cases} 3 \cdots 3^N + x = 13 \cdots 3^m \\ 3 \cdots 3^N - x = 5^m. \end{cases}$$

First, consider the case where. Adding these two equation yields:

$$(5 \cdot 13 \cdots 3)^m + 1 = 2 \cdot 3 \cdots 3^N,$$

which has only the solution $(m, N) = (1, 1)$ by Lemma 2.2. Hence equation has only the solution $(x, m, n) = (3 \cdots 32, 1, 2) = (3R_k - 1, 1, 2)$.

Next consider this case. Adding these two equation yields:

$$5^m + 13 \cdots 3^m = 2 \cdot 3 \cdots 3^N.$$

If m is even, then the equation modulo 3 implies that $2 \equiv 0 \pmod 3$, which is impossible. If m is odd, then the equation modulo one of the prime factors of the left side implies that the left side is divisible by the prime number, but the right side is not divisible by the prime number, which is a contradiction. Hence these completes the proof of Theorem 1.1 (iii). Therefore this completes the proof of Theorem 1.1. □

4 Remark

The values of k and result of prime factorization of $6R_k - 1$ satisfying the condition of Theorem 1.1 with $1 \leq k \leq 100$ obtained by using the Computational Algebra System MAGMA [BoCa] are given in the following table:

k	result of prime factorization of $6R_k - 1$
1	5
2	$5 \cdot 13$
16	$5 \cdot 1333333333333333$
42	$5 \cdot 133333333333333333333333333333333333333333$
84	$5 \cdot 13333333333333333333333333333333333333333333333333$ $3333333333333333333333333333333333$
96	$5 \cdot 13333333333333333333333333333333333333333333333333$ $3333333333333333333333333333333333333333333333$

Acknowledgment. The second author is supported by the Next Generation Research Encouragement Program of Nakamura Gakuen University.

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