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# On Topological Semi-Manifolds and their applications

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**Abstract**

In this paper, we provide a topological semi-manifold and its differential structure, its tangent space, and the definition of a topological semi-manifold with boundary to find a way to draw a map of the earth with high accuracy.

## 1 Introduction

The topic of manifolds has been raised since the seventies of the last century. Brickell [1] and Guillemin [2] defined a manifold as a topological space that achieves several properties; namely, that the space must be Hausdorff and local Euclidean of dimension $n$, and that this space must have a countable base. In 2022, Jasem and Tawfiq [3] presented a new type of manifolds called Bc manifold. As for the topic of continuous functions and equivalence functions, many researchers have presented studies on this type of functions,

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including Jasem and Tawfiq [4] who presented a new type of continuous functions. In this research, we present a new type of manifolds, called Topological semi-manifolds followed by topological semi-manifolds with boundaries with the aim of using them in drawing maps more accurately by converting each three-dimensional shape into a two-dimensional Euclidean space.

# 2 Topological semi-manifolds

**2.1 Definition** A topological space $\mathcal{M}^m$ is said to be a topological semi-manifold of dimension $m$ iff the following conditions are met:
1. $\mathcal{M}^m$ is semi-Hausdorff
2. For all $p \in \mathcal{M}^m$ there is a semi-open neighborhood $u$ such that $p \in u \subseteq \mathcal{M}^m$ and it is semi-homeomorphic to a semi-open set $\nu$ in $\mathcal{R}^m$
3. $\mathcal{M}^m$ has a countable base.
If $\eta : u \to \nu$ is a semi-homomorphism, then $(u,\eta)$ is called a coordinate neighborhood on the semi-manifold $\mathcal{M}^m$ at a point $p$, $\eta(p) = (\eta_1(p), \ldots\ldots, \eta_m(p))$, $(\eta_1, \ldots.., \eta_m)$ is called coordinate function on $u$ and $(\eta_1(p), \ldots\ldots, \eta_m(p))$ is called a local coordinate.

**2.2 Definition** Let $\mathcal{M}^m$ be a topological semi-manifold of dimension $m$ and let $\eta : u \to \nu$ be a semi-homeomorphism of a semi-open subset $u \subseteq \mathcal{M}^m$ onto the semi-open subset $\nu$ in $\mathcal{R}^m$. Then $\eta$ is called a chart of $\mathcal{M}^m$.

**2.3 Definition** The differentiable structure $\mathbb{Q}$ of class $C^k$ on a topological semi-manifold $\mathcal{M}^m$ is the set of all coordinate neighborhoods $(u_i,\eta_i)$ on $\mathcal{M}^m$; i.e.,
1. $\bigcup u_i = \mathcal{M}^m$
2. If $(u_i, \eta_i)$, $(u_j,\eta_j) \in \mathbb{Q}$ such that $u_i \cap u_j \neq \varnothing$, then the function
$\eta_i \circ \eta_j^{-1}$: $\eta_j(u_i \cap u_j) \to R^m$ is differentiable of class $C^k$
3. $\mathbb{Q} = \{(u_i,\eta_i)\}$ is maximal relative to condition 2.
If $\eta_i \circ \eta_j^{-1}$ are of class $C^\infty$ for all $i, j$, then we get the differentiable structure of class $C^\infty$.

**2.4 Definition** The topological semi-manifold with differentiable structure $\mathbb{Q}$ of class $C^k$ is called a smooth semi-manifold iff $\mathbb{Q}$ is of class $C^\infty$

**2.5 Theorem** Every manifold is a semi-manifold.

**Proof.** Let $A^m$ be a manifold. Then it is a Hausdorff space. That is, for all $x, y \in A^m$ there exist two disjoint open sets $u$,$\nu$ such that $x \in u, y \in \nu$. Since every open set is semi-open, $u$,$\nu$ are semi-open and $A^m$ is a semi-Hausdorff space.
Every point in $A^m$ has a neighborhood which is homeomorphic to an open subset of $R^m$. Since every continuous function is semi-continuous, every point in $A^m$ has a neighborhood which is semi-homeomorphic to a semi-open subset of $R^m$ and so $A^m$is semi-manifold.

**2.6 Remark** The converse of the above theorem is not necessarily true. That is, not every semi-manifold is a manifold as the following example shows:

**2.7 Example** Suppose $a, b \in R$. Let $A = \{x \in R : a < x \leq b\}$. Then $A$ is a semi-manifold topological space since every element in $A$ has a semi-open set which is homeomorphic to semi-open subset of $R$ and $A$ is semi-Hausdorff and has a countable base, However, it is not a manifold because we cannot find an open neighborhood for the element $b$.

**2.8 Lemma** There is a neighborhood $\upsilon$ of $a$ and $w$ of $\beta(a)$ such that $\beta : \upsilon \longrightarrow w$ is invertible with a smooth inverse.

**2.9. Theorem** Let $u$ be semi-open in $R^{n+m}$ and $\gamma : \ u \longrightarrow R^m$ be a $C^\infty$ function. Suppose that for each $a \in \gamma^{-1}(c)$ such that $c \in R^m$ and the derivative $D\gamma_a : \ R^{m+n} \longrightarrow R^m$ is surjective. Then $\gamma^{-1}(c)$ has the structure of an $n$-dimensional semi-manifold which is semi-Hausdorff and has a countable base of a semi-open set .

**Proof.** Suppose that the derivative $\gamma$ at $a$ is $D\gamma_a : \ R^{m+n} \longrightarrow R^m$ such that $\gamma(a) + D\gamma_a(h) + \mathcal{R}(a, h)$ where $\mathcal{R}(a, h)/\|h\| \ \longrightarrow 0 \ as \ h \longrightarrow 0$
$\gamma\,(x_1 \ldots\ldots, x_{n+m}) = (\gamma_1, \ldots, \gamma_m)$
$\frac{\partial \gamma_i}{\partial x_i}(a)$ $1 \leq i \leq m$ , $1 \leq j \leq n+m$
Now we get that this is surjective. By rearranging $x_1, \ldots\ldots x_{m+n}$, we may assume $\frac{\partial \gamma_i}{\partial x_i}(a)$, $1 \leq i \leq m$ , $1 \leq j \leq m$ is invertible. Now, define
$\beta : \upsilon \longrightarrow R^{n+m}$ such that
$\beta\,(x_1, \ldots, x_{n+m}) = (\gamma_1, \ldots\ldots, \gamma_m, x_{m+1}, \ldots, x_{n+m}) \ldots .(1)$
$\beta_a$ is invertible. By lemma 2.8 and (1), we show that $\beta$ maps $\upsilon \cap \gamma^{-1}(c)$ to the intersection of $w$ with $\{x \in R^{n+m} : x_i = c_i, 1 \leq i \leq m\}$.

This is a coordinate chart $\phi$. If we take two charts $\phi_{\alpha_1}$, $\phi_{\alpha_2}$, then $\phi_{\alpha_1} \circ \phi_{\alpha_2}^{-1}$ is a map from a semi-open set in $\{x \in R^{n+m} : x_i = c_1,\ 1 \leq i \leq m\}$ to another on which is the restriction of $\beta_{\alpha_1}(\beta_{\alpha_2}^{-1})$ of a semi-open in $R^{n+m}$ to itself, there is an invertible map $C^\infty$ in the induced topology from $R^{n+m}$, $\beta_\alpha$ is a semi-homeomorphism.
So semi open in the semi-manifold is the same as semi open in the induced topology, since$R^{n+m}$ is semi-Hausdorff with a countable basis of a semi-open set and so is $\gamma^{-1}(c)$ .

**2.10 Definition** Let $\mathcal{M}^m$ and $\mathbb{N}^m$ be smooth semi-manifolds. Then $f$ : $\mathcal{M}^m \longrightarrow \mathbb{N}^m$ is called a semi-diffeomorphism iff $f$ is a semi-homeomorphism and both $f$ and $f^{-1}$ are differentiable functions of class $C^\infty$. In this case, we call $\mathcal{M}^m$ and $\mathbb{N}^m$ diffeomorphic.

**2.11 Example** The function $\gamma$ : $R \longrightarrow R$ such that $\gamma(x) = x$ is a semi-homeomorphism since $\gamma$ , $\gamma^{-1}$ are differentiable functions of class $C^\infty$.

**2.12 Definition** Let $\mathcal{M} \subset R^k$ be a smooth semi-manifold and let $x \in \mathcal{M}$ be any point in $\mathcal{M}$. Since $\mathcal{M}$ is semi manifold, there is a parametrization $\gamma$ : $u \longrightarrow W \cap \mathcal{M}$ where $u$ is a semi-open subset of $R^k$ if $a \in u$ with $\gamma$ $(u) = x$, then $D\gamma_a : R^m \longrightarrow R^k$ is the derivative of $\gamma$ at $a$.

**2.13 Definition** The tangent of a semi-manifold $\mathcal{M}$ at a point $x$ is defined to be the image $(D\gamma_a\,(R^m))$ of $\gamma_a$ and is denoted by $T\mathcal{M}_x$; that is, $D\gamma_a\,(R^m) = T\mathcal{M}_x$.

**2.14 Theorem** Let $\mathcal{M}$ be a smooth semi-manifold. Then for all $x\in \mathcal{M}$ the dimension of $T\mathcal{M}_x$ is equal to the dimension of $\mathcal{M}$.

**Proof.** Let $m$ be the dimension of the semi-manifold $\mathcal{M}$ and let $x \in M \subset R^k$. Choose a neighborhood $u$ of a point $x$ in $\mathcal{M}$. Let $\beta$ : $u \longrightarrow \beta(u) \subset R^m$such that $\beta$ is a semi-diffeomorphism. Then $\beta^{-1} : \beta(u) \longrightarrow R^k$ $D\beta^{-1}$ is a linear map from a subset of $R^m$ into $R^k$. As a result, dim($T\mathcal{M}_x$) is at most $m$.
$\beta\ \circ \beta^{-1}$: $R^m \longrightarrow R^m$ is the identity, by the chain rule.
$\Rightarrow$Id=$D_{\beta(x)}\,(\beta \circ \beta^{-1}) = (D_x\beta) \circ (D_{\beta(x)}\beta^{-1})\Rightarrow$ the rank of $D_{\beta(x)}\beta^{-1}$ cannot be less than rank Id=$m$. Consequently, dim($T\mathcal{M}_x$) =dim($\mathcal{M}$).
(Note that for all $x, y\in \mathcal{M}$ such that $x \neq y$, we have $T\mathcal{M}_x \neq T\mathcal{M}_y$).

# 3 Semi-manifold with a boundary

**3.1 Definition** A subset $\mathcal{M} \subset R^k$ is an $n$-dimensional smooth semi-manifold with boundary if $\forall x \in \mathcal{M}$ there is a semi-open neighborhood $u$ of $x$ which is semi-diffeomorphic to a semi-open set $\nu \subseteq {R_+}^n$. The boundary of $\mathcal{M}$, denoted by $b(\mathcal{M})$, consists of all points that are mapped to $b\left({R_+}^n\right) \cong R^{n-1}$ under these diffeomorphisms.
Note that if $\mathcal{M}$ is an $n$-dimensional semi-manifold, then $b(\mathcal{M})$ is a smooth $(n-1)$-semi-manifold by just taking the restrictions of the charts to each boundary component and noticing that they are semi-diffeomorphisms between
$b\left(\mathcal{M}\right)$ and $\left(b{R_+}^n\right) \cong R^{n-1}$.

**3.2 Remark** $b(\mathcal{M})$ is a smooth semi-manifold without boundary; i.e., $b^2\left(\mathcal{M}\right) = b\left(b\left(\mathcal{M}\right)\right) = \varphi$.

**3.3 Example** Let $A = \{(x, y) \in R : x^2 + y^2 \leq 1\}$ be a semi-manifold with boundary. Then $b(A) = \{(x, y) \in R : x^2 + y^2 = 1\ \}$
Since $b(\mathcal{M})$ is an $(n-1)$-dimensional semi-manifold, for each boundary point $x \in b(\mathcal{M})$, there is a neighborhood $u \subset \mathcal{M}$ and a semi-diffeomorphism $\lambda : u \longrightarrow \upsilon \bigcap R^n_+$ where $\upsilon \subset R^n$ is semi-open, $\lambda^{-1}$ is defined on $\upsilon \bigcap R^n_+$ and is semi-diffeomorphism. Thus, there exists a semi-open set $\upsilon' \subseteq R^n$ and a smooth map $\psi : \upsilon' \longrightarrow R^k$ such that, $\psi\ \ / \ \upsilon \bigcap R^n_+ = \lambda^{-1}$.
From this, we define the tangent space:

**3.4 Definition** For a boundary point $x \in b(\mathcal{M})$, the tangent space to $\mathcal{M}$ at $x$ is the image of the differential of $\psi$; that is, $T_x\mathcal{M} = \{D_{\lambda(x)}\psi(y) : y \in \upsilon'\}$.

**3.5 Definition** A map $f : \mathcal{M} \rightarrow \mathbb{N}$ between smooth semi-manifolds with boundary is said to be smooth at $a$ point $x \in \mathcal{M}$ in the usual way if $x \in b(\mathcal{M})$. $f$ is smooth at $x$ if for every choice of semi-open sets $x \in\ \sqcap \subset \mathcal{M}$ and $f(x) \in \upsilon \subset \mathbb{N}$ and diffeomorphisms $\lambda : u \longrightarrow \lambda(u) \subset R^m$ and $\mu : \upsilon \longrightarrow \mu(\upsilon) \subset R^n$ there are extensions $\widetilde{\lambda}^{-1}$ and $\widetilde{\mu}$ of $\lambda^{-1}$and $\mu$ such that the composition: $\widetilde{\mu}\ o\ f\ o\widetilde{\lambda}^{-1} : W \subset R^m \longrightarrow Y \subset R^n$ on some neighborhoods $x \in W$ , $f(x) \in Y$ is smooth.

**3.6 Example** Let $E = \{(x, y, z) \in R^3 : x^2 + y^2 + z^2 \leq a^2\}$ such that $a \in R_+$. We verify that $E$ is smooth semi-manifold with boundary:

$E$ is a Hausdorff space, every point in $E$ has a semi-open set which is semi-homeomorphic to a semi-open subset in $R^2$.

$$u_1 = \{(x,y,z) \in E : z \geq 0\},\ u_2 = \{(x,y,z) \in E : z \leq 0\}$$

$$u_3 = \{(x,y,z) \in E : y \geq 0\},\ u_4 = \{(x,y,z) \in E : y \leq 0\}$$

$$u_5 = \{(x,y,z) \in E : x \geq 0\},\ u_6 = \{(x,y,z) \in E : x \leq 0\}$$

Let $\upsilon_1 = \upsilon_2 = \{(x,y) \in R^2 : x^2+y^2 \leq a^2\}$ $\upsilon_3 = \upsilon_4 = \{(x,z) \in R^2 : x^2+z^2 \leq a^2\}$ $\upsilon_5 = \upsilon_6 = \{(x,z) \in R^2 : x^2+z^2 \leq a^2\}$. $\upsilon_i$ with $1 \leq i \leq 6$ are semi-open subsets in $R^2$. Let $\eta_1 : u_1 \longrightarrow \upsilon_1; \eta_1(x,y,z) = [x,y]$, $\eta_2 : u_2 \longrightarrow \upsilon_2; \eta_2(x,y,z) = [x,y]$

$$\eta_3 : u_3 \longrightarrow \upsilon_3; \eta_3(x,y,z) = [x,z], \quad \eta_4 : u_4 \longrightarrow \upsilon_4; \eta_4(x,y,z) = [x,z]$$

$$\eta_5 : u_5 \longrightarrow \upsilon_5; \eta_5(x,y,z) = [y,z], \quad \eta_6 : u_6 \longrightarrow \upsilon_6; \eta_6(x,y,z) = [y,z]$$

. $\eta_i$ with $1 \leq i \leq 6$ are semi-homeomorphic. Now, let $p \in E$. Then $p = (x,y,z)$ such that $x^2+y^2+z^2 \leq a^2$. Either $x \neq 0$, $y \neq 0$, or $z \neq 0$. Let $x \neq 0$. If $x \geq 0$, then $p \in u_5$. If $x \leq 0$, then $p \in u_6$. Similarly, each $p \in E$ implies that there exist $u_i, 1 \leq i \leq 6$ such that $p \in u_i$ which is semi-homeomorphic to $\upsilon_i$ in $R^2$

Now, to show that $E$ has a countable base, let $\omega$ be the collection of all spheres whose centers are rational numbers. Thus $\omega$ is a countable base.

Let $\overline{\omega} = \{w \cap E : w \in \omega\}$. Then $\overline{\omega}$ is a countable basis for $E$.

Let $\mathbb{Q} = \{(u_1,\eta_1),(u_2,\eta_2),\ldots\ldots(u_6,\eta_6)\}$. The collection $u_i$ ; $1 \leq i \leq 6$ is a cover of $E$; that is, $\bigcup_{i=1}^{6} u_i = E$. Choose $(u_1,\eta_1),(u_5,\eta_1)$ where $\eta_1 : u_1 \longrightarrow \upsilon_1$ with $\eta_1(x,y,z) = [x,y]$, $\eta_5 : u_5 \longrightarrow \upsilon_5$ with $\eta_5(x,y,z) = [y,z]$, $u_1 \cap u_5 = \{(x,y,z) \in E : z \geq 0, x \geq 0\}$,$\eta_5(u_1 \cap u_5 = \{(x,z) \in R^2 : z \geq 0\}$.

To prove $\eta_1 \circ \eta_5^{-1} : \eta_5(u_1 \cap u_5) \to R^2$ is differentiable. Let $(y,z) \in \eta_5(u_1 \cap u_5) \to \eta_1 \circ \eta_5^{-1}(y,z) = \eta_1\left(\eta_5^{-1}(y,z)\right) = \eta_1(\sqrt{a^2-y^2-z^2}$,y,z)= $[\sqrt{a^2-y^2-z^2}$,y] which is differentiable of class $C^\infty$.

We conclude from this that $E$ is a semi-smooth manifold with boundary. Now, if we use (3.2) from which we conclude that $b(E)$ is also smooth semi-manifold $b(E) = \{(x,y,z) \in R^3 : x^2+y^2+z^2 = a^2\}$ and this represents the earth whose center is $(0,0,0)$. That is, it is a smooth semi-manifold. So, the map of the earth can be drawn accurately by transferring each point on the surface of the earth to a point on the Euclidean space with dimension 2 by using the homeomorphism function.

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