International Journal of Mathematics and Computer Science
Volume **20,** Issue no. 2, (2025), 685–687
DOI: https://doi.org/10.69793/ijmcs/02.2025/prugsapitak

On the Diophantine equations of the form $x^2 - kxy + y^2 + kx = 0$ and $x^2 - kxy + y^2 + 2kx = 0$

Supawadee Prugsapitak, Nattaporn Thongngam

Division of Computational Science
Faculty of Science
Prince of Songkla University
Hatyai, Songkhla, Thailand

email: supawadee.p@psu.ac.th, 6510230021@psu.ac.th

(Received March 4, 2025, Accepted April 3, 2025,
Published April 9, 2025)

Abstract

For an integer k, define $S(k)$ as the set of integers l where the equation $x^2 - lxy + y^2 + kx = 0$ has infinitely many positive integer solutions. We demonstrate that for positive integer k, $k \in S(k)$ if and only if $k = 3, 4, 5$; and for $k \equiv 0, 1, 3 \pmod 4$, $k \in S(2k)$ if and only if $k = 3, 4, 5$.

1 Introduction

For an integer k, let $S(k)$ be the set of integers l such that the equation $x^2 - lxy + y^2 + kx = 0$ has infinitely many positive integer solutions. In 2004, Marlewski and Zarzycki [4] proved that $S(1) = \{3\}$ and for any positive integer l, $l \notin S(1)$ for $l > 3$. In 2010, Keskin [3] showed that $S(5) = \{3\}$. In 2011, Yuan and Hu [5] determined $S(2) = \{3, 4\}$ and $S(4) = \{3, 4, 6\}$. Later, in 2013, Feng, Yuan, and Hu [2] confirmed that $S(k)$ is finite and provided $S(k)$ for $1 \le k \le 33$. In 2015, Cipu presented the following theorem [1]:

Theorem 1.1. *[1] Let k be a positive integer. Then, for any $a \in S(k)$, one has either $a = k + 2$ or $3 \le a \le \lfloor \frac{k+5}{2} \rfloor$. Moreover, $3, \lfloor \frac{k+5}{2} \rfloor$, and $k + 2$ belong to $S(k)$.*

Key words and phrases: Diophantine Equations, Quadratic Diophantine Equations, Infinitely Many Solutions.
AMS (MOS) Subject Classifications: 11D09, 11D99.
ISSN 1814-0432, 2025, https://future-in-tech.net

In this paper, we prove that, for any positive integer k, $k \in S(k)$ if and only if $k = 3, 4, 5$. Also, for a positive integer $k \equiv 0, 1, 3 \mod 4$, we prove that $k \in S(2k)$ if and only if $k = 3, 4, 5$.

2 Main results

Utilizing the results obtained from Theorem 1.1, we demonstrate that $k \in S(k)$ if and only if $k = 3, 4$ or 5.

Theorem 2.1. *Let k be a positive integer. The equation*

$$x^2 - kxy + y^2 + kx = 0 \tag{2.1}$$

has an infinite number of positive integer solutions x and y if and only if $k = 3, 4, 5$.

Proof. Let k be a positive integer and $k \in S(k)$. According to Theorem (1.1), the following inequality holds: $3 \leq k \leq \lfloor \frac{k+5}{2} \rfloor$.
Case 1. If k is even, then $3 \leq k \leq \frac{k}{2} + 2$. From this inequality, we conclude that $k = 4$.
Case 2. Now, if k is an odd integer, we have $3 \leq k \leq \frac{k+1}{2} + 2$. This leads us to the conclusion that k can be either 3 or 5.
For the converse, by Theorem 1.1 we obtain that $3 \in S(3)$, $4 = \lfloor \frac{4+5}{2} \rfloor \in S(4)$ and $5 = \lfloor \frac{5+5}{2} \rfloor \in S(5)$. □

Next, we consider the equation $x^2 - kxy + y^2 + 2kx = 0$.

Theorem 2.2. *Let k be a positive integer. If $k \equiv 0, 1, 3 \mod 4$, then the Diophantine equation*

$$x^2 - kxy + y^2 + 2kx = 0 \tag{2.2}$$

has an infinite number of positive integer solutions x and y if and only if $k = 3, 4, 5$.

Proof. Suppose that positive integers x, y, k satisfy the equation $x^2 - kxy + y^2 + 2kx = 0$. Thus, $x^2 - kxy + y^2$ is even. Obviously, x and y have the same parity. We next show that x and y are even. For an odd integer k, since $x^2 - kxy + y^2$ is even, x and y must be even. For $k \equiv 0 \mod 4$, if x and y are both odd, then we have $x^2 - kxy + y^2 + 2kx \equiv 2 \mod 4$ which is a contradiction. Hence, for any $k \equiv 0, 1, 3 \mod 4$, both x and y are even.

Thus, we assume $x = 2x'$ and $y = 2y'$ for some positive integers x' and y'. Then, we have $(2x')^2 - k(2x')(2y') + (2y')^2 + 2k(2x') = 0$. Upon dividing by 4, we derive the equation $x'^2 - kx'y' + y'^2 + kx' = 0$. By Theorem 2.1, it follows that k is $3, 4$ or 5. $\square$

For $k = 2$, we prove the following:

Theorem 2.3. *The Diophantine equation*

$$x^2 - 2xy + y^2 + 4x = 0 \tag{2.3}$$

has no positive integer solutions.

Proof. Given $x^2 - 2xy + y^2 + 4x = 0$, it follows that $(x - y)^2 = -4x$. As x is a positive integer, this is not possible. $\square$

Considering an integer k where $k > 2$ and $k \equiv 2 \pmod 4$, it becomes intriguing to examine the conditions under which the equation $x^2 - kxy + y^2 + 2kx = 0$ has infinitely many solutions in positive integers. We identify this as an unresolved problem and encourage additional study in this area.

Acknowledgment. We thank the referee for valuable comments and suggestions.

References

[1] M. Cipu, Quadratic Diophantine Equation with Infinitely Many Solutions in Positive Integers, Integers, **15,** (2015).

[2] L. Feng, P. Yuan, Y. Hu, On the Diophantine equation $x^2 - kxy + y^2 + lx = 0$, Integers, **13,** (2013).

[3] R. Keskin, Solutions of some quadratic Diophantine equations, Computers and Mathematics with Applications, **60,** (2010), 2225–2230.

[4] A. Marlewski, P. Zarzycki, Infinitely many positive solutions of Diophantine equation $x^2 - kxy + y^2 + x = 0$, Computers and Mathematics with Applications, **47,** (2004), 115–121.

[5] P. Yuan, Y. Hu, On the Diophantine equation $x^2 - kxy + y^2 + lx = 0, l \in \{1, 2, 4\}$, Computers and Mathematics with Applications, **61,** (2011), 573–577.