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Generalizations of a bound for the rational functions with relaxing zeros

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Abstract

We give an upper bound for the modulus of rational functions and show how it generalizes some well-known inequalities.

1 Introduction

In this paper, we use $T_k = \{z \in \mathbb{C} : |z| = k\}, D_k^+ = \{z \in \mathbb{C} : |z| > k\}$ where k is any positive integer. Let $w(z) := \prod_{j=1}^{n}(z - a_j)$ and $B(z) := \prod_{j=1}^{n}\left(\frac{1-\overline{a_j}z}{z-a_j}\right) = \frac{w^*(z)}{w(z)}$, where $a_j \in \mathbb{C}$ for $j = 1, 2, 3, \ldots, n$ and $w^*(z) = z^n\overline{w\left(\frac{1}{\overline{z}}\right)}$. The product $B(z)$ is called Blaschke product when $|B(z)| = 1$ for $z \in T_1$. Let P_m be the class of all polynomials of degree at most m. We define $R_{m,n}$ by $R_{m,n} = R_{m,n}(a_1, a_2, \ldots, a_n) := \left\{\frac{p(z)}{w(z)} : p \in P_m \text{ and } m \le n\right\}$, where

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$a_j \in D_1^+, j = 1, 2, \ldots, n$ and $m \le n$. Therefore, $R_{m,n}$ is the set of all rational functions with poles $a_1, \ldots, a_n$ at most and with finite limit at ∞. In 2022, Gulzar, Zargar, and Akhter [1] investigated an upper bound for the modulus of rational functions.

Theorem 1.1. *If $r(z) \in R_{n,n}$, has a zero of order ν at z_0 with $|z_0| < 1$ and the remaining $n - \nu$ zeros lie in $T_1 \cup D_1^+$, then for $z \in T_1$*

$$\max_{z \in T_1} |r'(z)| \le \frac{1}{2} \left(\frac{1 + |z_0|}{1 - |z_0|} \right)^{\nu} \left[|B'(z)| + \nu \left(\frac{1 - |z_0|}{1 + |z_0|} \right) \right] \max_{z \in T_1} |r(z)|.$$

Then, Gupta, Hans, and Mir [2] proved the following Theorem:

Theorem 1.2. *If $r(z) \in R_{m,n}$, has two zeros of order μ and ν at z_0 and z_1 respectively with $|z_0| < 1, |z_1| < 1$ and the remaining $m - \mu - \nu$ zeros lie in $T_1 \cup D_1^+$, then for $z \in T_1$*

$$\max_{z \in T_1} |r'(z)| \le \frac{1}{2} \left(\frac{1 + |z_0|}{1 - |z_0|} \right)^{\mu} \left(\frac{1 + |z_1|}{1 - |z_1|} \right)^{\nu} \left[|B'(z)| - (n - m) + \mu \left(\frac{1 - |z_0|}{1 + |z_0|} \right) \right.$$
$$\left. + \nu \left(\frac{1 - |z_1|}{1 + |z_1|} \right) \right] \max_{z \in T_1} |r(z)|.$$

2 Main Results

In this paper, we generalize some inequalities for the modulus of rational functions. Let us start with a lemma due to Rasri and Phanwan [4].

Lemma 2.1. *If $r(z) \in R_{m,n}$ and all the zeros of $r(z)$ lie in $T_k \cup D_k^+, k \ge 1$, then for $z \in T_1$*

$$|r'(z)| \le \frac{1}{2} \Big[|B'(z)| + \frac{2m - n(1 + k)}{1 + k} \Big] |r(z)|.$$

Theorem 2.2. *Assume $r(z) \in R_{m,n}$ has the zero z_0 of order s_0 with $|z_0| < k$ and the remaining $m - s_0$ zeros lie in $T_k \cup D_k^+, k \ge 1$. Then for $z \in T_1$*

$$\max_{z \in T_1} |r'(z)| \le \frac{1}{2} \left(\frac{1 + |z_0|}{|1 - |z_0||} \right)^{s_0} \left[|B'(z)| + \frac{2(m - s_0) - n(1 + k)}{1 + k} + \frac{2s_0}{1 + |z_0|} \right] \max_{z \in T_1} |r(z)|.$$

Proof. Let $r(z) = (z - z_0)^{s_0} h(z)$ where $h(z)$ is a rational function having all zeros lying in $T_k \cup D_k^+, k \geq 1$. Differentiation $r(z)$ with respect to z, we get

$$r'(z) = (z - z_0)^{s_0} h'(z) + s_0 h(z)(z - z_0)^{s_0 - 1}.$$

Triangle inequality implies that

$$\max_{z \in T_1} |r'(z)| \leq (1 + |z_0|)^{s_0} \max_{z \in T_1} |h'(z)| + s_0(1 + |z_0|)^{s_0 - 1} \max_{z \in T_1} |h(z)| \qquad (2.1)$$

for $z \in T_1$. Applying Lemma 2.1 for $|h'(z)|$, the inequality (2.1) yields

$$\max_{z \in T_1} |r'(z)| \leq (1 + |z_0|)^{s_0} \Big[\frac{1}{2}\Big(|B'(z)| + \frac{2(m - s_0) - n(1+k)}{1+k}\Big)\Big] \max_{z \in T_1} |h(z)| \\ + s_0(1 + |z_0|)^{s_0 - 1} \max_{z \in T_1} |h(z)|$$

for $z \in T_1$. Since $\max_{z \in T_1} |h(z)| \leq \frac{\max_{z \in T_1} |r(z)|}{|1 - |z_0||^{s_0}}$, we get

$$\max_{z \in T_1} |r'(z)| \leq \frac{1}{2} \left(\frac{1 + |z_0|}{|1 - |z_0||}\right)^{s_0} \left[|B'(z)| + \frac{2(m - s_0) - n(1+k)}{1+k} + \frac{2s_0}{1 + |z_0|}\right] \max_{z \in T_1} |r(z)|$$

for $z \in T_1$. The proof is complete. □

Remark 2.3. 1. *By letting $k = 1, m = n$ in Theorem 2.2, it reduces to Theorem 1.1.*

2. *By letting $k = 1$ in Theorem 2.2, it reduces to Corollary 2.7 of Gupta, Hans, and Mir [2]*

3. *By letting $s_0 = 0$ and $k = 1$ in Theorem 2.2, it reduces to Corollary 12 of Rasri and Phanwan [3].*

Theorem 2.4. *Assume $r(z) \in R_{m,n}$ has the zero z_0 of order s_0 and the zero z_1 of order s_1 with $|z_0| < k, |z_1| < k$ and the remaining $m - (s_0 + s_1)$ zeros lie in $T_k \cup D_k^+, k \geq 1$. Then for $z \in T_1$*

$$\max_{z \in T_1} |r'(z)| \leq \frac{1}{2} \left[\left(\frac{1 + |z_0|}{|1 - |z_0||}\right)^{s_0} \left(\frac{1 + |z_1|}{|1 - |z_1||}\right)^{s_1} \left(|B'(z)| + \frac{2(m - s_0 - s_1) - n(1+k)}{1+k} + \frac{2s_0}{1 + |z_0|}\right) \\ + \frac{2s_1}{1 + |z_1|} \left(\frac{1 + |z_1|}{|1 - |z_1||}\right)^{s_1}\right] \max_{z \in T_1} |r(z)|.$$

Proof. Let $r(z) = (z - z_1)^{s_1} r_0(z)$, where $r_0(z) = \frac{(z-z_0)^{s_0} h(z)}{w(z)}$ and $h(z)$ be a rational function having all its zeros lying in $T_k \cup D_k^+, k \geq 1$. Differentiating $r(z)$ with respect to z, we obtain $r'(z) = (z-z_1)^{s_1} r_0'(z) + s_1 r_0(z)(z-z_1)^{s_1-1}$. The Triangle Inequality implies that

$$\max_{z\in T_1} |r'(z)| \leq (1+|z_1|)^{s_1} \max_{z\in T_1} |r_0'(z)| + s_1(1+|z_1|)^{s_1-1} \max_{z\in T_1} |r_0(z)|,$$

for $z \in T_1$. Applying Theorem 2.2 for $|r_0'(z)|$, we get

$$\max_{z\in T_1} |r'(z)| \leq \Bigg[\frac{(1+|z_0|)^{s_0}(1+|z_1|)^{s_1}}{2|1-|z_0||^{s_0}} \left(|B'(z)| + \frac{2(m-s_0-s_1)-n(1+k)}{1+k} + \frac{2s_0}{1+|z_0|}\right) + s_1(1+|z_1|)^{s_1-1}\Bigg] \max_{z\in T_1} |r_0(z)|,$$

for $z \in T_1$. Since $\max_{z\in T_1} |r_0(z)| \leq \frac{\max_{z\in T_1}|r(z)|}{|1-|z_1||^{s_1}}$, we get

$$\max_{z\in T_1} |r'(z)| \leq \frac{1}{2}\Bigg[\left(\frac{1+|z_0|}{|1-|z_0||}\right)^{s_0}\left(\frac{1+|z_1|}{|1-|z_1||}\right)^{s_1}\left(|B'(z)| + \frac{2(m-s_0-s_1)-n(1+k)}{1+k} + \frac{2s_0}{1+|z_0|}\right) + \frac{2s_1}{1+|z_1|}\left(\frac{1+|z_1|}{|1-|z_1||}\right)^{s_1}\Bigg] \max_{z\in T_1} |r(z)|$$

for $z \in T_1$. Thus the proof is complete. □

Remark 2.5. *By letting $k = 1$ in Theorem 2.4, it reduces to Theorem 1.2.*

Theorem 2.6. *Assume $r_v(z) = \frac{(z-z_v)^{s_v}(z-z_{v-1})^{s_{v-1}}\cdots(z-z_0)^{s_0}h(z)}{w(z)} \in R_{m,n}$, where $r_v(z)$ has the zeros $z_0, z_1, \ldots, z_v$ with $|z_i| < k, k \geq 1$ for $0 \leq i \leq v$ and $h(z)$ has all its zeros lying in $T_k \cup D_k^+, k \geq 1$. Then, for $0 \leq i \leq v$ and $z \in T_1$*

$$\begin{aligned}\max_{z\in T_1} |r_v'(z)| \leq \frac{1}{2}\Bigg[&\left(\frac{1+|z_0|}{|1-|z_0|}\right)^{s_0}\left(\frac{1+|z_1|}{|1-|z_1|}\right)^{s_1}\cdots\left(\frac{1+|z_i|}{|1-|z_i|}\right)^{s_i}\Bigg(|B'(z)| \\ &+\frac{2(m-s_0-s_1-\cdots-s_i)-n(1+k)}{1+k} + \frac{s_0}{1+|z_0|}\Bigg) \\ &+\frac{2s_1}{1+|z_1|}\left(\frac{1+|z_1|}{|1-|z_1|}\right)^{s_1}\cdots\left(\frac{1+|z_i|}{|1-|z_i|}\right)^{s_i} \\ &+\frac{2s_2}{1+|z_2|}\left(\frac{1+|z_2|}{|1-|z_2|}\right)^{s_2}\cdots\left(\frac{1+|z_i|}{|1-|z_i|}\right)^{s_i} + \cdots \\ &+\frac{2s_i}{1+|z_i|}\left(\frac{1+|z_i|}{|1-|z_i|}\right)^{s_i}\Bigg] \max_{z\in T_1} |r_v(z)|. \end{aligned} \tag{2.2}$$

Proof. Let $r_v(z) = \frac{(z-z_v)^{s_v}(z-z_{v-1})^{s_{v-1}}\cdots(z-z_0)^{s_0}h(z)}{w(z)}$, where $r_v(z)$ has the zeros $z_0, z_1, \dots, z_v$ with $|z_i| < k$ for $0 \le i \le v$ and the remaining $n-(s_0+s_1+\cdots+s_v)$ zeros lie in $T_k \cup D_k^+, k \ge 1$. Let $r_0(z) = (z-z_0)^{s_0}\frac{h(z)}{w(z)}$ and $r_1(z) = (z-z_1)^{s_1} r_0(z)$. An upper bound of $|r_0'(z)|$ is obtained by Theorem 2.2. Using the fact that $|r_0(z)| \le \frac{|r_1(z)|}{|1-|z_1||^{s_1}}$, we get an upper bound of $|r_1(z)|$ as in Theorem 2.4. Let $r_i(z) = (z-z_i)^{s_i} r_{i-1}$. We can find an upper bound of $|r_i'(z)|$ for $1 \le i \le v$ by a similar process using an upper bound of $|r_{i-1}'(z)|$ from the previous process and the fact that $|r_{i-1}(z)| \le \frac{|r_i(z)|}{||1-|z_i||^{s_i}}$ for $1 \le i \le v$. Finally, we get inequality (2.2). $\square$

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