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# Injective Coloring of the Powers of Cycles

**Panuvit Chuaephon, Kittikorn Nakprasit**

Department of Mathematics
Faculty of Science
Khon Kaen University
Khon Kaen 40002, Thailand

email: panuvitc@kkumail.com, kitnak@kku.ac.th

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**Abstract**

In this work, we study the injective chromatic number of the graph $C_n^k$, which is a graph obtained from $C_n$ by adding an edge to any two distinct vertices with distance 2 to $k$ on $C_n$. An injective coloring of a graph $G$ is a vertex coloring in which any pair of vertices with a common neighbor are assigned different colors. The injective chromatic number $\chi_i(G)$ of a graph $G$ is the least positive integer $k$ such that $G$ has an injective $k$-coloring. In this work, we determine $\chi_i(C_n^k)$ for all $n$ and $k$. When $n \leq 2k+1$, the graph $C_n^k$ becomes a complete graph $K_n$, with injective chromatic number $n$. For larger values of $n$, we obtain the following main results: if $r=0$, then $\chi_i(C_n^k) = 2k+1$, while if $1 \leq r \leq 2k$, then $\chi_i(C_n^k) = 2k + \tilde{x} + 1$, where $\tilde{x}$ is the largest natural number such that $1 + (2k+1)\lceil \frac{n}{2k+x} \rceil > n$. Additionally, we find $\chi_i(P_n^k)$ for all $n$ and $k$ where $P_n^k$ is a graph obtained similarly from a path $P_n$.

# 1 Introduction

Imagine a network of communication stations such that each station has a single communication channel, and any two stations may or may not communicate with each other directly. To prevent confusion, we require that no

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two stations with the same channel can communicate directly with the same station. For example, if stations A and B have the same channel, they cannot both communicate directly with station C.

This problem of assigning channels to the stations can be modeled as an injective coloring problem in graph theory, where the stations are represented by vertices, direct communications between two stations are represented by edges, and the communication channels are represented by colors. The goal is to find the minimum number of colors (or channels) needed to assign to the vertices so that two vertices with the same neighbor receive distinct colors. The problem becomes finding the injective chromatic number of a specific graph which we define and discuss in detail later in this section.

Throughout this study, we consider $G$ as a simple, finite, and undirected graph with the vertex set $V(G)$. A vertex coloring of $G$ is said to be an *injective coloring* if any pair of vertices with a common neighbor are assigned different colors. Note that such a coloring is not necessarily proper, meaning that adjacent vertices may be colored with the same color. The *injective chromatic number* $\chi_i(G)$ of $G$ is the least positive integer $k$ needed for an injective $k$-coloring of $G$.

In 2002, G. Hahn et al. [4] defined the injective coloring of a graph and gave some bounds of the injective chromatic number in general, plus some exact values. Additionally, they provided a characterization for graphs in which each bound is attained. Furthermore, they presented the results on the injective chromatic numbers of Cartesian product of graphs, including hypercubes.

In 2008, Hell et al. [5] demonstrated that for any chordal graph $G$, the computation of $\chi_i(G)$ is at least as efficient as determining $\chi(G-B)^2$, where $\chi(G-B)^2$ is the chromatic number of $(G-B)^2$ and $B$ is the set of bridges of $G$. This finding indicates that the injective chromatic number can be computed in polynomial time for both strongly chordal graphs and so-called power chordal graphs.

In 2010, Cranston et al. [3] proved several results regarding the injective chromatic numbers. In particular, they showed that if the maximum average degree of $G$ (mad$(G)$) is at most $\frac{5}{2}$, then $\chi_i(G) \leq \Delta(G)+1$. Furthermore, if $\text{mad}(G) < \frac{42}{19}$, then $\chi_i(G) = \Delta(G)$. They also show that, for planar graphs $G$ with minimum cycle length (girth) of $g(G)$ and $\Delta(G) \geq 4$, if $g(G) \geq 9$, then $\chi_i(G) \leq \Delta(G)+1$. Additionally, if $g(G) \geq 13$, then $\chi_i(G) = \Delta(G)$.

In 2015, Song and Yue [8] gave sharp bounds, and in some cases exact values, for the injective chromatic numbers of the Cartesian product, join, union, direct product, lexicographic product, and disjunction of graphs.

In this work, we focus on the injective chromatic number of a specific graph, $C_n^k$.

If the network has stations arranged in a circular manner such that each station can communicate directly to any other station within a certain distance, the resulting network can be modeled using $C_n^k$, the $k$-th power of a cycle.

The graph $C_n^k$ has been the subject of extensive study in graph theory. Several works have investigated various types of graph coloring on $C_n^k$, including list coloring [7], total coloring [2], incidence coloring [6], acyclic coloring [1], and star coloring [1].

This work aims to determine $\chi_i(C_n^k)$ for all $n$ in terms of integers $k, t$, and $r$, where $n = (2k+1)t + r$ with $t \geq 1$ and $0 \leq r \leq 2k$. To the best of our knowledge, exact results of $\chi_i(C_n^k)$ have not been provided in the literature for all $n$ and $k$.

Our main theorem directly addresses this gap in the literature by providing explicit formula for $\chi_i(C_n^k)$ as follows. When $n \leq 2k+1$, the graph $C_n^k$ becomes a complete graph $K_n$, with injective chromatic number $n$. For larger values of $n$, we obtain the following main results: if $r = 0$, then $\chi_i(C_n^k) = 2k+1$, while if $1 \leq r \leq 2k$, then $\chi_i(C_n^k) = 2k + \tilde{x} + 1$, where $\tilde{x}$ is the largest natural number such that $1 + (2k+1)\lceil \frac{n}{2k+x} \rceil > n$.

## 2 Main results

First, we determine $\chi_i(P_n^k)$, for $n \geq 3$ and $k \geq 2$.

**Theorem 2.1.** $\chi_i(P_n^k) = \begin{cases} 2k+1, & \textit{if } n \geq 2k+1, \\ 2k+1-d, & \textit{if } n = 2k+1-d \textit{ where } 1 \leq d \leq 2k-2. \end{cases}$

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the vertices of $P_n$ in the usual arrangement.

For the case $n \geq 2k+1$, we define the color set to be $\{1, 2, \ldots, 2k+1\}$. Now, we give the coloring of $P_n^k$ as follows. For the first $2k+1$ vertices, we color these vertices as $1, 2, \ldots, 2k+1$. Observe that any two colored vertices have a common neighbor, and thus $\chi_i(P_n^k) \geq 2k+1$. Next, we give colors $1, 2, \ldots, 2k+1, \ldots, 1, 2, \ldots$ consecutively for $n-2k-1$ remaining vertices. One can see that this is a $(2k+1)$-injective coloring. Thus we have $\chi_i(P_n^k) \leq 2k+1$, and this case is complete. For the case $n = 2k+1-d$ where $1 \leq d \leq 2k-2$, the upper bound $\chi_i(P_{2k+1-d}^k) \leq n = 2k+1-d$ follows immediately. Since any two vertices of $P_{2k+1-d}^k$ have a common neighbor, it follows that $\chi_i(P_{2k+1-d}^k) \geq 2k+1-d$. This completes the proof.

Now we proceed to find $\chi_i(C_n^k)$. Let $v_1, v_2, \ldots, v_n$ be the vertices of $C_n$ in the usual arrangement. Observe that if $n \leq 2k+1$, then $C_n^k$ forms a complete graph $K_n$. The injective chromatic number of this graph is $n$ as shown in [4] for $n \geq 3$. Therefore it remains to consider the case where $n \geq 2k+2$ for $C_n^k$.

From this point forward, we express $n$ in the form $n = (2k+1)t + r$ where $t$ and $r$ are integers satisfying $t \geq 1$ and $0 \leq r \leq 2k$.

**Lemma 2.2.** *If $n = (2k+\alpha+\beta)b + (2k+\alpha)d$ where $\alpha, b$ are positive integers and $\beta, d$ are nonnegative integers, then $\chi_i(C_n^k) \leq 2k + \alpha + \beta$.*

**Proof.** Assume $n$ as in the lemma. To establish the bound for $\chi_i(C_n^k)$, we give a coloring as follows.

For the first $(2k + \alpha + \beta)b$ vertices, we divide these vertices into $b$ blocks of consecutive $2k + \alpha + \beta$ vertices and we give a coloring of each block as $1, 2, \ldots, 2k + \alpha + \beta$.

For the remaining $(2k + \alpha)d$ vertices, we divide these vertices into $d$ blocks of consecutive $2k + \alpha$ vertices and we give a coloring of each block as $1, 2, \ldots, 2k + \alpha$. Observe that any two vertices with the same color lie in distinct blocks and thus they have no common neighbor. This yields $\chi_i(C_n^k) \leq 2k + \alpha + \beta$ as desired.

**Lemma 2.3.** *If an injective coloring on $C_n^k$ yields a color class of size $b$, then $n \geq (2k + 1)b$.*

**Proof.** Consider a color class $A$ of size $b$. Since any pair of vertices in $A$ have no common neighbors, there are at least $2k$ vertices in $C_n$ between any two vertices in $C_n$. It follows that $n \geq (2k + 1)b$ as desired.

**Theorem 2.4.** *$\chi_i(C_n^k) = 2k + 1$ if and only if $n$ is divisible by $2k + 1$.*

**Proof.** *Necessity.* Let $n = (2k + 1)t + r$ such that $t \geq 1$ and $1 \leq r \leq 2k$. Suppose to the contrary that $C_n^k$ has an injective coloring using $2k+1$ colors. By the Pigeonhole principle, there are $\lceil \frac{n}{2k+1} \rceil = t + 1$ vertices of the same color. It follows from Lemma 2.3 that $n \geq (2k+1)(t+1) = (2k+1)t+(2k+1)$, a contradiction. Thus $\chi_i(C_n^k) > 2k + 1$.

*Sufficiency.* Let $n = (2k + 1)t$. It follows from Lemma 2.2 that $\chi_i(C_n^k) \leq 2k + 1$. Since the graph $C_n^k$ contains $P_n^k$ as a subgraph, it follows from Theorem 2.1 that $\chi_i(C_n^k) \geq 2k + 1$. Thus $\chi_i(C_n^k) = 2k + 1$.

Recall that $n = (2k+1)t+r$. Now, we assume $r \geq 1$ and let $\tilde{x}$ be the largest natural number such that $1 + (2k+1)\lceil \frac{n}{2k+\tilde{x}} \rceil > n$. Since $1 + (2k+1)\lceil \frac{n}{2k+1} \rceil = 1 + (2k + 1)(t + 1) = (2k + 1)t + (2k + 2) > n$, it follows that $\tilde{x} \geq 1$.

**Lemma 2.5.** *If $r \geq 1$, then $\chi_i(C_n^k) \geq 2k + \tilde{x} + 1$.*

**Proof.** Let $r \geq 1$, and suppose to the contrary that $C_n^k$ has an injective coloring using $2k + \tilde{x}$ colors. By the Pigeonhole principle, there are $\lceil \frac{n}{2k+\tilde{x}} \rceil$ vertices with the same color. Since $\lceil \frac{n}{2k+\tilde{x}} \rceil \geq t + 1$ by the definition of $\tilde{x}$, it follows from Lemma 2.3 that $n \geq (2k+1)\lceil \frac{n}{2k+\tilde{x}} \rceil = (2k+1)(t+1)$. However, this contradicts $n = (2k+1)t + r$ where $r \leq 2k$. Thus $\chi_i(C_n^k) \geq 2k + \tilde{x} + 1$.

**Lemma 2.6.** *If $1 \leq r \leq t$, then $\tilde{x} = 1$ and $\chi_i(C_n^k) = 2k + 2$.*

**Proof.** Let $1 \leq r \leq t$. From the proof of Theorem 2.4, we have $\chi_i(C_n^k) \geq 2k + 2$. Using Lemma 2.2 on $n = (2k+1)t + r = (2k+2)r + (2k+1)(t-r)$, we have $\chi_i(C_n^k) \leq 2k + 2$. Thus $\chi_i(C_n^k) = 2k + 2$.

To show $\tilde{x} = 1$, we write $n = (2k + 2)t - (t - r)$. Consider $\lceil \frac{n}{2k+2} \rceil = t - \lfloor \frac{t-r}{2k+2} \rfloor \leq t$. Consequently, $1 + (2k + 1)\lceil \frac{n}{2k+2} \rceil \leq 1 + (2k + 1)t \leq n$. By definition of $\tilde{x}$, we have $\tilde{x} < 2$. Recall that $\tilde{x} \geq 1$. Thus $\tilde{x} = 1$ as desired.

We now present the main theorem of this paper.

**Theorem 2.7.** *If $r \geq 1$, then $\chi_i(C_n^k) = 2k + \tilde{x} + 1$, where $\tilde{x}$ is the largest natural number such that $1 + (2k + 1)\lceil \frac{n}{2k+x} \rceil > n$.*

**Proof.** It follows from Lemma 2.6 that the theorem is true for $r \leq t$. Now let $1 \leq t < r \leq 2k$. It follows from Lemma 2.5 that $\chi_i(C_n^k) \geq 2k + \tilde{x} + 1$.

To establish $\chi_i(C_n^k) \leq 2k+\tilde{x}+1$, note that $n = (2k+\tilde{x})t+r-(\tilde{x}-1)t = (2k+\tilde{x}+1)(r-(\tilde{x}-1)t)+(2k+\tilde{x})(t-r+(\tilde{x}-1)t)$. If $r-(\tilde{x}-1)t \geq 1$ and $t-r+(\tilde{x}-1)t \geq 0$, then Lemma 2.2 yields that $\chi_i(C_n^k) \leq 2k+\tilde{x}+1$. Thus it remains to show that $r - (\tilde{x} - 1)t \geq 1$ and $t - r + (\tilde{x} - 1)t \geq 0$.

From the definition of $\tilde{x}$, we have $1 + (2k+1)\lceil \frac{n}{2k+\tilde{x}} \rceil > n \geq 1 + (2k+1)t$. Since $n = (2k + \tilde{x})t + r - (\tilde{x} - 1)t$, we have $(2k + 1) \left( t + \lceil \frac{r-(\tilde{x}-1)t}{2k+\tilde{x}} \rceil \right) \geq (2k + 1)t + 1$. Thus $r - (\tilde{x} - 1)t \geq 1$ as required.

From the definition of $\tilde{x}$ again, we have $1 + (2k + 1)\lceil \frac{n}{2k+\tilde{x}+1} \rceil \leq n$. Since $n = (2k + \tilde{x} + 1)t + r - \tilde{x}t$, it follows that $(2k + 1)t + r = n \geq 1 + (2k + 1)\lceil \frac{n}{2k+\tilde{x}+1} \rceil = 1 + (2k + 1)t + (2k + 1)\lceil \frac{r-\tilde{x}t}{2k+\tilde{x}+1} \rceil$.

Consequently, $r - 1 \geq (2k + 1)\lceil \frac{r-\tilde{x}t}{2k+\tilde{x}+1} \rceil$. Since $2k \geq r$, it follows that $r - \tilde{x}t \leq 0$. Thus $t - r + (\tilde{x} - 1)t \geq 0$ as required, completing the proof.

# 3 Conclusion

In conclusion, we establish the explicit formula for $\chi_i(C_n^k)$ where $n = (2k+1)t + r$ in the usual form as follows:

- If $t = 0$, that is $n \leq 2k+1$, then $\chi_i(C_n^k) = n$;
- If $t \geq 1$ and $r = 0$, then $\chi_i(C_n^k) = 2k+1$;
- If $t \geq 1$ and $1 \leq r \leq 2k$, then $\chi_i(C_n^k) = 2k + \tilde{x} + 1$, where $\tilde{x}$ is the largest natural number such that $1 + (2k+1)\lceil \frac{n}{2k+x} \rceil > n$.

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