International Journal of Mathematics and Computer Science
Volume **20,** Issue no. 2, (2025), 481–488
DOI: https://doi.org/10.69793/ijmcs/02.2025/muh

A Notion of Angle using a 2-Semi-inner Product on the Space of p-Summable Sequences

Muh Nur[1], **Al Azhary Masta**[2], **Mawardi Bahri**[1], **Anna Islamiyati**[3]

[1]Department of Mathematics
Faculty of Mathematics and Natural Science
Hasanuddin University
Makassar, Indonesia

[2]Mathematics Study Program
Faculty of Mathematics and Science Education
Universitas Pendidikan Indonesia
Bandung, Indonesia

[3]Department of Statistics
Faculty of Mathematics and Natural Science
Hasanuddin University
Makassar, Indonesia

email: muhammadnur@unhas.ac.id

(Received December 4, 2024, Accepted January 5, 2025, Published January 6, 2025)

Abstract

We introduce a 2-semi-inner product on the space of p-summable sequences. Using this 2-semi-inner product, we define the h_p-orthogonality and the h_p-angle between two vectors and discuss their properties. Moreover, we formulate the h_p-angle between two 2-dimensional subspaces that intersects a 1-dimensional subspace. Using this formula, we construct the space of p-summable sequences as a strictly convex 2-normed space.

Key words and phrases: 2-semi-inner product, h_p-orthogonality, h_p-angle, p-summable sequences, strictly convex.
AMS (MOS) Subject Classifications: 15A03, 46B20, 46E30.
ISSN 1814-0432, 2025, https://future-in-tech.net

1 Introduction

The theory of 2-normed spaces was initially introduced by Gähler [3] in the mid 1960's and that of 2-inner product spaces was initially introduced by Diminnie, Gähler and [4] in 1970's. Since then, many researchers have studied these two spaces and obtained various results ([5, 6, 7, **?**]. In a 2-inner product space $(X, \langle\cdot,\cdot|\cdot\rangle)$, we can calculate angles between two subspaces. Using the standard 2-inner product and the standard 2-norm in inner product space, the concept of the angle between two subspace in general has been studied intensively — see [8, 9, **?**]. Then, Nur et al. [10, 11, 12] introduced the angle between two subspaces in a normed space that is not an inner product space. As it is known, not all 2-normed spaces are 2-inner product spaces. For instance, the space ℓ^p for $1 \leq p < \infty$ and $p \neq 2$, equipped with the 2-norm $\|\cdot,\cdot\|_p$, defined by Gunawan [6]:

$$\|x,y\|_p = \left[\frac{1}{2}\sum_j\sum_k \left(\text{abs}\begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix}\right)^p\right]^{\frac{1}{p}} \tag{1.1}$$

is not a 2-inner product space.
Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. The functional $h : X^3 \to R$ defined by $h(x,y)(z) = \frac{1}{2}\|x,y\|\,[\tau_+(x,y)(z) + \tau_-(x,y)(z)]$ with $\tau_\pm(x,y)(z) = \lim_{t\to\pm 0}\frac{\|x+tz,y\|-\|x,y\|}{t}$, satisfies the following properties:

1. $h(x,y)(x) = \|x,y\|^2$ for every $x, y \in X$;
2. $h(\alpha x,y)(\beta z) = \alpha\beta h(x,y)(z)$ for every $x,y,z \in X$ and $\alpha,\beta \in \mathbb{R}$;
3. $h(x,y)(x+z) = \|x,y\|^2 + h(x,y)(z)$ for every $x,y,z \in X$;
4. $|h(x,y)(z)| \leq \|x,y\|\,.\,\|y,z\|$ for every $x,y,z \in X$.

If, in addition, the functional $h(x,y)(z)$ is linear in terms of yz, then h is called *a 2-semi-inner product* on X [1]. Note that, in $(X, \langle\cdot,\cdot|\cdot\rangle)$, the functional $h(x,y)(z)$ is identical with $\langle x,y|\cdot\rangle$. Specifically, $(\ell^p, \|\cdot,\cdot\|_p)$ raises two questions:
Can we define an explicit form the 2-semi-inner product on ℓ^p? Can we define the ortogonality and the angle on ℓ^p with the equpped 2-norm?
In this article, we will introduce the angle between two subspace of a 2-normed space using the 2-semi-inner product h. We also apply this formula to examine a strictly convex 2-normed space. In the last section, we will

discuss the 2-semi-inner product $h(x, y)(z)$ in ℓ^p spaces. Moreover, ℓ_p space, equipped 2-norm, is strictly convex.

2 Main Result

2.1 A 2-semi-inner product on ℓ^p

In this subsection, we will discuss the angle between two subspaces of $(\ell^p, \|\cdot,\cdot\|_p)$. We first state the definition 2-semi inner product of ℓ^p spaces. Let $(\ell^p, \|\cdot,\cdot\|_p)$ be a 2-normed space with $1 \le p < \infty$. Take a set $\{x, y, z\}$ in ℓ^p. We define the following mapping $h_p(x, z)(y)$ by

$$h_p(x,y)(z) = \frac{\|x,y\|_p^{2-p}}{2} \sum_j \sum_k \left(\text{abs} \begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix} \right)^{p-1} \left(\text{sgn} \begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix} \right) \begin{vmatrix} z_j & z_k \\ y_j & y_k \end{vmatrix}. \tag{2.2}$$

Then we have the following result.

Theorem 2.1. *The mapping $h_p(x, y)(z)$ in (2.2) defines a 2-semi-inner product on ℓ^p.*

Proof. We will verify that $h_p(x, y)(z)$ satisfies the four properties of functional h and linear in terms of z.

1. Observe that $h_p(x, y)(x) = \|x, y\|_p^2$.

2. By using the properties of determinants, we have

$$h_p(ax, y)(bz) = ab h_p(x, y)(z).$$

3. Observe that

$$\begin{aligned} h_p(x,y)(x+z) &= \left[\frac{\|x,y\|_p^{2-p}}{2} \sum_j \sum_k \left(\text{abs} \begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix} \right)^{p-1} \right. \\ &\quad \left. \text{sgn} \begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix} \left(\begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix} + \begin{vmatrix} z_j & z_k \\ y_j & y_k \end{vmatrix} \right) \right] \\ &= \|x,y\|_p + h_p(x,y)(y). \end{aligned}$$

4. Observe that

$$\begin{aligned}
|h_p(x,y)(z)| &\leq \frac{\|x,y\|_p^{2-p}}{2}\sum_j\sum_k\left(\text{abs}\begin{vmatrix} x_j & x_k \\ y_j & y_k\end{vmatrix}\right)^{p-1}\left(\text{abs}\begin{vmatrix} z_j & z_k \\ y_j & y_k\end{vmatrix}\right)\\
&\leq \|x,y\|_p^{2-p}\left[\left(\frac{1}{2}\sum_j\sum_k\left(\text{abs}\begin{vmatrix} x_j & x_k \\ y_j & y_k\end{vmatrix}\right)^p\right)^{\frac{p-1}{p}}\right.\\
&\qquad\left.\left(\frac{1}{2}\sum_j\sum_k\left(\text{abs}\begin{vmatrix} z_j & z_k \\ y_j & y_k\end{vmatrix}\right)^p\right)^{\frac{1}{p}}\right]\\
&= \|x,y\|_p\,\|y,z\|_p\,.
\end{aligned}$$

5. Using the properties of determinants,
   $h_p(x,y)(z+z') = h_p(x,y)(z) + h_p(x,y)(z')$.

Therefore, $h_p(x,y)(z)$ defines a 2-semi-inner product on ℓ^p. □

Remark 2.2. *Note that, for $p = 2$, we have $h_2(x,x)(z) = h_2(z,z)(x)$ and $h_2(x,y)(z) = h_2(y,x)(z)$. Hence, $h_2(x,y)(z)$ is the 2-inner product on ℓ^2.*

2.2 h_p-Orthogonality and h_p-angle

Let $(\ell^p, \|\cdot,\cdot\|_p)$ be a 2-normed space and let $\{a_1, a_2\}$ be a linearly independent set in ℓ^p. First, we write the norm defined by Gunawan in [6] as follows: $\|x\|_{h_p} := \left[\|x,a_1\|_p^2 + \|x,a_2\|_p^2\right]^{\frac{1}{2}}$, for every $x \in \ell^p$. Next, we can define a mapping that is obtained from the 2-semi inner product $h(\cdot,\cdot)(\cdot)$ by

$$[x,z]_{h_p} = h_p(x,a_1)(z) + h_p(x,a_2)(z) \tag{2.3}$$

where $x, z \in \ell^p$. Then, we obtain the proposition as follows:

Proposition 2.3. *The mapping $[x,z]_{h_p}$ in (2.3) defines a semi-inner product on $(\ell^p, \|\cdot,\cdot\|_p)$.*

Proof.
1. Using the properties of 2-semi inner product, we have $[x,x]_{h_p} \geq 0$ and $[x,x]_{h_p} = 0$ if and only if $x = 0$.

2. Using the properties of 2-semi inner product, we have $[\alpha x, \beta z]_{h_p} = h_p(\alpha x, a_1)(\beta z) + h(\alpha x, a_2)(\beta z) = \alpha\beta h_p(x,a_1)(z) + \alpha\beta h_p(x,a_2)(z) = \alpha\beta[x,z]_{h_p}$.

3. We observe that $[x, y+z]_{h_p} = h_p(x, a_1)(y+z) + h_p(x, a_2)(y+z) = h_p(x, a_1)(y) + h_p(x, a_1)(z) + h_p(x, a_2)(y) + h_p(x, a_2)(z) = [x, y]_{h_p} + [x, z]_{h_p}$.

4. We observe that $\left|[x, z]_{h_p}\right| \leq |h_p(x, a_1)(z)| + |h_p(x, a_2)(z)|$. Using the properties of 2-semi inner product, we have $\left|[x, z]_{h_p}\right| \leq \|x, a_1\|_p \|z, a_1\|_p + \|x, a_2\|_p \|z, a_2\|_p$. Since $2ab \leq a^2 + b^2$ for any $a, b \geq 0$, we obtain $\left([x, z]_{h_p}\right)^2 \leq (\|x, a_1\|_p^2 + \|x, a_2\|_p^2)(\|z, a_1\|_p^2 + \|z, a_2\|_p^2) = \|x\|_{h_p}^2 \|z\|_{h_p}^2$. □

Remark 2.4. *In general, the semi inner product $[\cdot, \cdot]_{h_p}$ does not satisfy the commutative property. For example, the space ℓ^1 with $a_1 = (1, 0, 0, \dots)$ and $a_2 = (0, 1, 0, \dots)$. Take $x = (1, 2, 2, 0, \dots)$ and $z = (3, 1, -2, 0, \dots)$. Clearly, $\|x, a_1\|_1 = 8$, $\|x, a_2\|_1 = 6$, $\|z, a_1\|_1 = 6$ and $\|z, a_2\|_1 = 10$. Therefore, $[x, z]_{h_1} = -2$ and $[z, x]_{h_1} = -10$. Hence $[x, z]_{h_p} \neq [z, x]_{h_p}$. If $p = 2$, then we can observe that $[x, z]_{h_2}$ is the inner product.*

Next, by using the semi-inner product $[\cdot, \cdot]_{h_p}$, we introduce h_p-orthogonality and he angle between two nonzero vectors x and z on $(\ell^p, \|\cdot, \cdot\|_p)$ as follows:

Definition 2.5. *Let $(\ell^p, \|\cdot, \cdot\|_p)$ be a 2-normed space. A vector x is h_p-*orthogonal *to z, and we symbolize as $x \perp_{h_p} z$, if and only if $[x, z]_{h_p} = 0$. Then, the angle between two nonzero vectors x and z is defined by $\Theta_{h_p}(x, z)$ such that*

$$\Theta_{h_p}(x, z) := \arccos \frac{[x, z]_{h_p}}{\|x\|_{h_p} \|z\|_{h_p}}.$$

Note that $\Theta_{h_p}(x, z) = \frac{1}{2}\pi$ if and only if $[x, z]_{h_p} = 0$ or $x \perp_{h_p} z$. Since $\left|[x, z]_{h_p}\right| \leq |x\|_{h_p} \|z\|_{h_p}$, $\pi \leq \Theta_{h_p}(x, z) \leq \pi$.

The angle $\Theta_{h_p}(\cdot, \cdot)$ has the following properties.

Proposition 2.6. *Let $(\ell^p, \|\cdot, \cdot\|_p)$ be a 2-normed space. The angle $\Theta_{h_p}(x, z)$ satisfies the following properties:*

(a) If x and z are of the same direction, then $\Theta_{h_p}(x, z) = 0$; if x and z are of the opposite direction, then $\Theta_{h_p}(x, z) = \pi$ (part of parallelism property).

(b) $\Theta_{h_p}(\alpha x, \beta z) = \Theta_{h_p}(x, z)$ if $\alpha\beta > 0$; $\Theta_{h_p}(\alpha x, \beta z) = \pi - \Theta_{h_p}(x, z)$ if $\alpha\beta < 0$ (homogeneity property).

(c) If $x_n \to x$ (in norm), then $\Theta_{h_p}(x_n, z) \to \Theta_{h_p}(x, z)$ (part of continuity property).

Proof. (a) Let $z = kx$ for an arbitrary nonzero vector x in ℓ^p and $k \in \mathbb{R}-\{0\}$. We have $\Theta_{h_p}(x,z) = \arccos \frac{[kx,x]_{h_p}}{\|x\|_{h_p}\|kx\|_{h_p}} = \arccos \frac{k\|x\|^2_{h_p}}{|k|\|x\|^2_{h_p}}$. Hence, $\Theta_{h_p}(x,z) = 0$ for $k > 0$ and $\Theta_{h_p}(x,z) = \pi$ for $k < 0$.

(b) Let $\alpha, \beta \in \mathbb{R}-\{0\}$. Observe that $\Theta_{h_p}(\alpha x, \beta z) = \arccos \frac{\alpha\beta[x,z]_{h_p}}{|\alpha\beta|\|x\|_{h_p}\|y\|_{h_p}}$.

(c) If $x_k \to x$ (in norm $\|\cdot\|_{h_p}$), then $|[x, z_n - z]_{h_p}| \le \|x\|_{h_p}\|z_n - z\|_{h_p} \to 0$. Observe that $[x, z_n - z]_{h_p} = [x, z_n]_{h_p} - [x, z]_{h_p}$. We have $[x, z_n]_{h_p} \to [x, z]_{h_p}$. Hence, $\Theta_{h_p}(x, z_n) \to \Theta_{h_p}(x, z)$,

as desired. □

Remark 2.7. *From Remark 2.4, we can conclude that the angle* $\Theta_{h_p}(\cdot,\cdot)$ *does not satisfy the symmetry property. Likewise, the g-angle does not satisfy the continuity property. For instance, take* $z_n = (\frac{1}{n}, 2, 0, \ldots)$, $z = (0, 2, 0, \ldots)$, *and* $x = (2, 2, 0, ..)$ *in* ℓ^1. *We observe that* $\cos\Theta_{h_1}(x, z_n) = 0$ *for any* $n \in \mathbb{N}$, *but* $\cos\Theta_{h_1}(x,z) \neq 0$.

3 Further Results

Using the 2-semi-inner product h, we define the angle between subspaces $U = \text{span}\{u,w\}$ and $V = \text{span}\{v,w\}$ of the 2-normed space, as follows.

Definition 3.1. *If* $U = span\{u,w\}$ *and* $V = span\{v,w\}$ *are 2-dimensional subspaces of* $(X, \|\cdot,\cdot\|)$ *that intersects on 1-dimensional subspace* $W = span\{w\}$, *then the angle between* U *and* V *is defined by* $\Theta_h(U,V)$ *with* $\cos\Theta_h(U,V) = \frac{h(u,w)(v)}{\|u,w\|\|v,w\|}$.

Remark 3.2. *Using Properties 4 in 2-semi-inner product* h, *we have* $-\pi \le \Theta_{h_p}(U,V) \le \pi$. *This fact shows that Definition 3.1 makes sense.*

For example, in ℓ^1, *take* $u = (3,1,0,\cdots)$, $v = (2,1,0,\cdots)$, *and* $w = (1,-1,0,\cdots)$. *Using formula in Definition 3.1, we obtain* $\Theta_{h_1}(U,V) = 0$.

Next, we will show $(\ell^p, \|\cdot,\cdot\|)$ is strictly convex 2-normed space. First, we recall definition of strictly convex 2-normed space as follows:

Definition 3.3. *[13] Let* $x, y \in (X, \|\cdot,\cdot\|)$ *be non-zero elements and let* $V(x,y)$ *denote the subspace of* X *generated by the vectors* x *and* y. *The space* X *is strictly convex if* $\|x,y\| = \|y,z\| = \left\|\frac{x+y}{2}, z\right\| = 1$ *and* $z \notin V(x,y)$, *for* $x, y, z \in X$, *imply* $x = y$.

Then Franić [2] give the connection between strictly convex 2-normed space and 2-semi-inner product in following theorem.

Theorem 3.4. *Let $[\cdot,\cdot|\cdot]$ be a 2-semi-inner product compatible with a 2-norm on $(X, \|\cdot,\cdot\|)$. X is strictly convex if and only if $[x, y|z] = \|x, z\|\,\|y, z\|$ and $z \notin V(x, y)$ for $x, y, z \in X$ implies $y = ax$, for some $a > 0$.*

Next, we obtain the following result.

Theorem 3.5. *Let $(\ell^p, \|\cdot,\cdot\|_p)$ be a 2-normed space where $1 \le p < \infty$ and $\Theta_{h_p}(U, V)$ be the angle between two subspaces $U = span\{u, w\}$ and $V = span\{v, w\}$ in ℓ^p. Then the following statements are equivalent:*
(1) ℓ^p is strictly convex.
(2) If $\cos\Theta_{h_p}(U, V) = 1$ where $w \notin span\{u, v\}$, then $u = av$ for some $a > 0$.

Proof. Let us first prove that (1) implies (2). Suppose ℓ^p is strictly convex. If $\cos\Theta_{h_p}(U, V) = 1$, $w \notin span\{u, v\}$ then $h_p(u, w)(v) = \|u, w\|_p\,\|v, w\|_p$,$w \notin span\{u, v\}$. Since $h_p(u, w)(v)$ is a 2-semi inner product on ℓ^p, then $u = av$ for some $a > 0$ by Theorem 3.4. Next, we show that (2) implies (1). Assume $\cos\theta_{h_p}(U, V) = 1$ where $w \notin span\{u, v\}$. Then $u = av$ for some $a > 0$. Suppose $h_p(u, w)(v) = \|u, w\|_p\,\|v, w\|_p$, $w \notin span\{u, v\}$. By Definition 2.2, we obtain $\cos\Theta_{h_p}(U, V) = 1$, $w \notin span\{u, v\}$ and by assumption, $u = av$, for some $a > 0$. Finally, we have $h_p(u, w)(v) = \|u, w\|_p\|v, w\|_p$, $w \notin span\{u, v\}$ implies $u = av$, for some $a > 0$. According to Theorem 3.4, ℓ^p is a strictly convex 2-normed space. □

4 Conclusion

In this work, we have formulated the the 2-semi-inner product h on ℓ^p spaces. Using this formula, we have defined the h_p-orthogonality and the h_p-angle between two vectors and have discussed theirs properties. Moreover, we have defined the h_p-angle between $U = span\{u, w\}$ and $V = span\{v, w\}$ in ℓ^p. We also have proved ℓ_p space as a strictly convex 2-normed space.

5 Acknowledgments

The research is supported by Fundamental Research Program 2024
No. 02035/UN4.22.2/PT.01.03/2024.

References

[1] K. Anevska, R. Malčeski, "Generalization on parallelepiped equality in 2-normed space", *Int. J. Pure Appl. Sci. Tech.,* **22**, no. 2, (2014), 41–49.

[2] I. Franič, "Two Results in 2-Normed Spaces", *Glas. Mat. Ser. III,* **17**, no. 37, (1982), 271–275.

[3] S. Gähler, "Lineare 2-normierte räume", *Math. Nachr.* **28,** (1964), 1–43.

[4] C. Diminnie, S. Gähler, A. White, "2-inner product spaces", *Demonstratio Math.* **6,** (1973), 525–536.

[5] C. Diminnie, S. Gähler, A. White, "Strictly convex linear 2-normed spaces", *Math. Nachr.,* **59,** (1974), 319–324.

[6] H. Gunawan, "The space of p-summable sequences and its natural n-norm", *Bull. Austral. Math. Soc.*. **64,** (2001) 137–147.

[7] H. Gunawan, M. Mashadi, "On finite-dimensional 2-normed space", *Soochow J. Math.*, **27,** (2001), 321–329.

[8] H. Gunawan, O. Neswan (2005), "On angles between subspaces of inner product spaces, *J. Math. Fund. Sci.,* **11,** (2005), 129–135.

[9] H. Gunawan, O. Neswan, W. Setya-Budhi, "A formula for angles between two subspaces of inner product spaces", *Beitr. Algebra Geom.* **46,** no. 2, (2005), 311–320.

[10] M. Nur, H. Gunawan, "A note on the g-angle between subspaces of a normed space", *Aequationes Math.,* **95,** (2021), 309–318.

[11] M. Nur, M. Bahri, A. Islamiyati, H. Batkunde, A new semi-inner product and p_n-angle in the space of p-summable sequences, *Mathematics*, **11,** (2023), 3139.

[12] M. Nur, M. Idris, Firman, "Angle in the space of p-summable sequences", *AIMS Mathematics,* **7,** no. 2, (2022), 2810–2819.

[13] A.H. Siddiqi, S.M. Rizwi, "2-Semi-inner spaces II", *Math. Japon*, **20,** (1976), 391–397.