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Hesitant Fuzzy Distributive R–Modules

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Abstract

Let R be a unital commutative ring. In this paper, we present and explore Hesitant Fuzzy Distributive R–Modules and Hesitant Fuzzy Chained R–Modules as generalizations of Fuzzy Distributive R–Modules and Fuzzy Chained R–Modules. Additionally, we give some fundamental properties of these generalized concepts.

1 Introduction

In this article, we introduce hesitant fuzzy distributive R–modules as a generalization of fuzzy distributive modules, which were presented by Hadi and Semeein [1]. We begin with a review of some definitions and necessary results. Next, we present several results and examples related to the concept of hesitant fuzzy distributive R–modules. We then investigate the relationship between hesitant fuzzy distributive R–modules and hesitant fuzzy chained

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R–modules [2]. Finally, this definition can be generalized in the broader fields of applied mathematics, particularly in domaination [3]–[4] or algebra [5]–[6].

Definition 1.1. *[1] Let D be an R–module and let F be a fuzzy module over D. F is called distributive if it satisfies the following condition:*

$$\gamma \cap (\chi + \mu) = (\gamma \cap \chi) + (\gamma \cap \mu)$$

for all fuzzy submodules γ, χ and μ of F.

Definition 1.2. *[1] Let F be a fuzzy set in H. For every $j \in [0, 1]$, the set $F_j = \{a \in H,\ F(a) \geq j\}$ is called the level subset of F.*

Remark 1.3. *[1] The following properties of the level subsets hold for each $j \in (0, 1]$:*
1) $(\gamma \quad \chi)_j = \gamma_j \quad \chi_j$
2) $\gamma = \chi$ if and only if $\gamma_j = \chi_j$, for all $j \in (0, 1]$.

Proposition 1.4. *[1] Let γ and χ be two fuzzy R–submodules of a given fuzzy R–module F. Then $(\gamma\chi)_j = \gamma_j\ \chi_j$, for every $j \in (0, 1]$.*

Definition 1.5. *[2] Let F be a hesitant fuzzy module over an R–module D. Then, F is said to be hesitant fuzzy chained R–module (HFCM) if, for every pair of hesitant fuzzy R–submodules γ and χ of F, one of the following holds:*

$$\gamma \subseteq \chi \text{ or } \chi \subseteq \gamma$$

2 Hesitant Fuzzy Distributive R–Modules

In this section, we generalize the concept of fuzzy distributive modules to Hesitant Fuzzy Distributive R–Modules and investigate some of their fundamental properties.

Definition 2.1. *Let D be an R–Module and let F be a Hesitant fuzzy module over D. D is called Distributive (HFDM) if, for any Hesitant fuzzy submodules γ, χ and μ of F, the following condition is satisfied:*

$$\gamma \cap (\chi + \mu) = (\gamma \cap \chi) + (\gamma \cap \mu)$$

Theorem 2.2. *A fuzzy module F of an R–module D is called a hesitant fuzzy distributive module if and only if F_j is a fuzzy distributive module, for all $j \in (0, 1]$.*

Proof. Let F be a hesitant fuzzy distributive R–module. To prove F_j is fuzzy distributive module for every $j \in (0, 1]$, let γ, χ and μ be submodules of F_j. Define:

$E(a) = \begin{cases} \{j\} & a \in \gamma \\ \{0\} & a \notin \gamma \end{cases}$, $Z(a) = \begin{cases} \{j\} & a \in \chi \\ \{0\} & a \notin \chi \end{cases}$, $H(a) = \begin{cases} \{j\} & a \in \mu \\ \{0\} & a \notin \mu \end{cases}$

It is clear that E, Z, H are hesitant fuzzy R–submodules of F and $E_j = \gamma$, $Z_j = \chi$, $H_j = \mu$. Since F is hesitant fuzzy distributive, $E \cap (Z\ H) = (E \cap Z) + (E \cap H)$. Hence, $[E \cap (Z + H)]_j = [(E \cap Z) + (E \cap H)]_j$, for all $j \in (0, 1]$. $E_j \cap (Z + H)_j = (E \cap Z)_j + (E \cap H)_j$ and $E_j \cap (Z_j + H_j) = E_j \cap Z_j) + (E_j \cap H_j)$ by Remark 1.3 and Proposition 1.4.
Thus, $\gamma \cap (\chi + \mu) = (\gamma \cap \chi) + (\gamma \cap \mu)$.
Conversely, if F_j is a fuzzy distributive R–module for every $j \in (0, 1]$. To prove F is a hesitant fuzzy distributive R–module. Let E, Z and H are hesitant fuzzy R–submodules of F. Then E_j, Z_j and H_j are fuzzy R–submodules of F_j, for every $j \in (0, 1]$. As F_j is a fuzzy distributive R–module, $E_j \cap (Z_j + H_j) = (E_j \cap Z_j) + (E_j \cap H_j)$. $E_j \cap (Z + H)_j = (E \cap Z)_j + (E \cap H)_j$ and $[E \cap (Z + H)]_j = [(E \cap Z) + (E \cap H)]_j$, by Remark 1.3 and Proposition 1.4. Then, $E \cap (Z + H) = (E \cap Z) + (E \cap H)$ by Remark 1.3. □

Example 2.3. *Let R be a ring and let D be a fuzzy R–module. Write $D = R \bigoplus R$ and let $F : D \longrightarrow [0, 1]$ be defined with $F(a) = 1$. Let*

$E(a, b) = \begin{cases} \{1\} & (a, b) \in R(1, 1) \\ \{0\} & otherwise \end{cases}$ $Z(a, b) = \begin{cases} \{1\} & (a, b) \in R(0, 1) \\ \{0\} & otherwise \end{cases}$,

$H(a, b) = \begin{cases} \{1\} & (a, b) \in R(1, 0) \\ \{0\} & otherwise \end{cases}$ $E_j = R(1, 1)$, $Z_j = R(0, 1)$, $H_j = R(1, 0)$

for all $j \in (0, 1]$. $E_j \cap (Z_j + H_j) = R(1, 1)$, $(E_j \cap Z_j) + (E_j \cap H_j) = (R(1, 1) \cap R(0, 1)) + (R(1, 1) \cap R(1, 0)) = 0 + 0 = 0$.
Then $E_j \cap (Z + H)_j \neq (E \cap Z)_j + (E \cap H)_j$ which implies F_j is not a fuzzy distributive R–module. Thus F is not a hesitant fuzzy distributive R–module.

Now, we will present the connection between the hesitant fuzzy distributive R-module and the hesitant fuzzy chained R-module.

Proposition 2.4. *Let F be a hesitant fuzzy chained R–module of an R–module D. Then, F is a hesitant fuzzy distributive R–module.*

Proof. Let E, Z, H hesitant fuzzy R–submodules of F. We may assume that $E \subseteq Z \subseteq H$. Therefore, $E \cap (Z + H) = (E \cap Z) = E\ (E \cap H)$, but $(E \cap Z) + (E \cap H) = E + E = E$. Thus, $E \cap (Z + H) = (E \cap Z) + (E \cap H)$. □

The converse of Proposition 2.4 is not true in general as illustrated by the following example:

Example 2.5. *Let $F(a) = 1$ for all $a \in \gamma, F_j = \gamma, \forall j \in [0,1]$. But γ is fuzzy distributive. Hence, by Theorem 2.2, F is a fuzzy distributive. However, F is not chained since there exist hesitant fuzzy R–submodules E, Z such that:*
$E(a) = \begin{cases} \{1\} & a \in 2\gamma \\ \{0\} & a \notin 2\gamma \end{cases}$, $\mathrm{Z}(a) = \begin{cases} \{1\} & a \in 3\gamma \\ \{0\} & a \notin 3\gamma \end{cases}$ *and* $E \nsubseteq \mathrm{Z} \nsubseteq E$.

Remark 2.6. *If $\gamma \leq \chi$ and χ is hesitant fuzzy distributive R–module, then γ is also a hesitant fuzzy distributive R–module.*

Proof. Let E, Z, H be hesitant fuzzy R–submodules of γ. Then E, Z, H hesitant fuzzy R–submodules of χ. Now, $\gamma \leq \chi$. But χ is hesitant fuzzy distributive R–module. Hence. $E \cap (\mathrm{Z} + H) = (E \cap \mathrm{Z}) + (E \cap H)$ which implies that γ is hesitant fuzzy distributive. ☐

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