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Innovative Perspectives on Antimagic Labeling in Graphs

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Abstract

The graph G represents an undirected, simple, finite graph. G's total labeling is a bijection between its vertex and edge sets and the set $\{1, 2, \ldots, p+q\}$, where p and q describe the cardinality of G's vertex and edge sets, respectively. In this paper, we explore the concept of Super Vertex Perfectly Total Antimagic (SVPTAT) labeling in the context of graph theory, specifically focusing on complete graphs and complete bipartite graphs $K_{m,n}$.

We present the conditions for graphs that do not confess vertex-magic and edge-antimagic at the same time. For any $n \geq 2$, the

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Corona product of any graph with mK_1 and mK_2 does not accept vertex-magic and edge-antimagic totals concurrently. Finally, we suggest a couple of open problems for future research.

1 Introduction

Graph labeling assigns integers to a graph's elements such as its vertices, edges, or both, subject to specific conditions. Total labeling is the term used to describe labeling in which the domain consists of vertices and edges. The total weight of an edge is the sum of the labels of an edge and its terminal vertices. The total labeling's vertex weight is calculated by summing the vertex label with the labels of all adjacent edges. Vertex-magic total labeling occurs when vertex weights are all constant. A vertex-magic total graph permits vertex-magic total labeling. If each edge has a distinct weight, the labeling is edge-antimagic total. An edge-antimagic total graph permits such labeling.

Researchers developed the notion of edge-antimagic total labeling [8]. In this paper, we focus on SVPTAT Antimagic Graphs, a particular class of antimagic graphs. Moreover, we establish conditions under which $K_{m,n}$ admits SVPTAT labeling for various cases based on the parity and relative sizes of m and n. Proofs are provided for these cases, alongside explicit labeling schemes. Some researchers have demonstrated that specific graph types possess the characteristics of both edge-magic total and vertex-antimagic total, or vertex-magic total and edge-antimagic total in [3]. For a comprehensive review of related studies and developments in this area, we refer the readers to [1, 2, 5, 6, 7, 9, 10, 12]. The concept of an antimagic total (TAT) graph was presented in [4]. If a label exhibits antimagic properties at both the edge and the vertex, it receives the designation of TAT. In [11], Swathi et al. introduced the notion of perfectly antimagic total (PAT) labeling. Perfectly antimagic total labels refer to antimagic labels, characterized by pairwise differences in vertex and edge weights.

The following open problem was proposed by Bača, et al. in [4].
Open problem Characterize the graphs that allow a total labeling that is simultaneously vertex-magic and edge-antimagic.

This paper looks at several graph families, such as path, cycle, and sun graphs, that exhibit vertex-magic and edge-antimagic total labelings.

2 Antimagic Graph Labeling through SVPTAT of K_n and $K_{m,n}$

This section explores labeling schemes for complete bipartite graphs,$K_{m,n}$, analyzing both even and odd cases of m and n, and extends to scenarios where $m \neq n$ or $m > n$.

Theorem 2.1. *For every positive integer $n \geq 3$, the complete graph K_n is SVPTAT.*

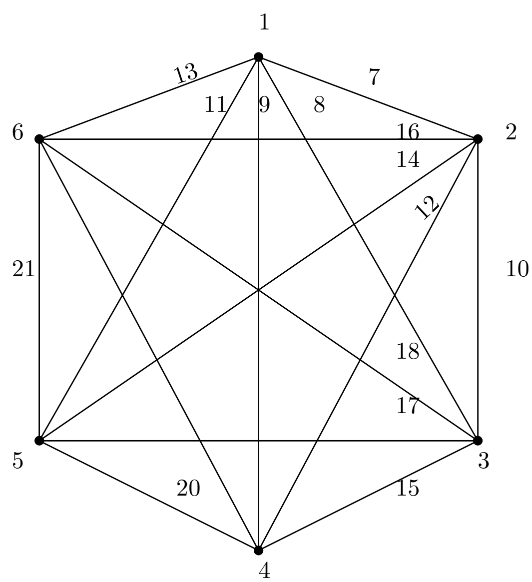

Figure 1: SVPTAT in K_6

Proof. The complete graph k_6 allows SVPTAT, as shown in Figure 1. K_n is SVPAT for $n = 3$. For $n \geq 4$, at every vertex in K_n, assign a label $1, 2, \ldots, n$. Under the vertex labeling established above, we now obtain the weight of all edges as $w(e_i) \leq w(e_j)$, for $1 \leq i < j \leq \frac{n(n-1)}{2}$. At this point, we assign s to e_i and t to e_j, given that $n + 1 \leq s < t \leq \frac{n(n+1)}{2}$. All complete graphs are SVPAT under this label. $\square$

Theorem 2.2. *The complete bipartite graph $K_{m,n}$ admits SVPTAT.*

Proof. Let v_i and u_i be the vertices of $K_{m,n}$ and whose edges are $\{v_i u_i; 1 \leq i \leq m \;\&\; 1 \leq j \leq n\}$ respectively.
Case (i) $m = n = t$ is even & $m > n$, $m \neq n$ and if $m > 2$ is even and $n > 3$

is odd

For each $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$

$$\begin{aligned} f(v_i) &= i \\ f(u_j) &= t + j \\ f(v_i u_1) &= m + n + i \\ f(v_i u_2) &= 2m + n + i \\ f(v_i u_3) &= 3m + n + i \\ f(v_i u_j) &= jm + n + i \end{aligned}$$

The complete bipartite is $K_{4,4}$ shown in Figure 2. Vertex weight of v_i,

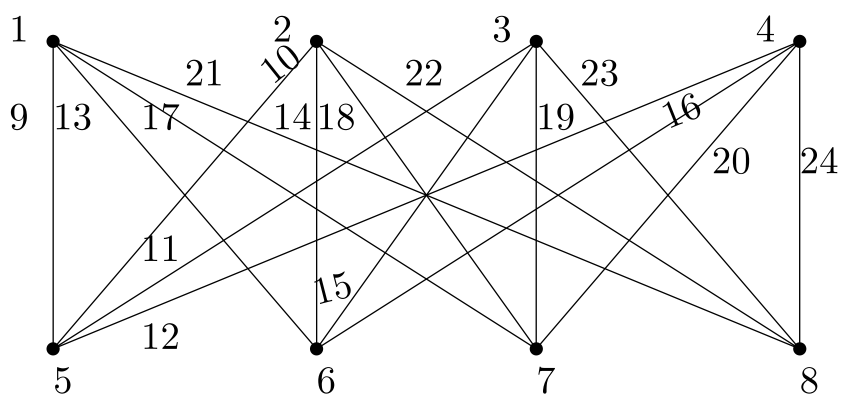

Figure 2: SVPTAT in $K_{4,4}$

$$\begin{aligned} wt_f(v_i) &= f(v_i) + \sum_{j=1}^{n} f(v_i u_j) \\ &= i + m + n + i + 2m + n + i + 3m + n + i \\ ... &+ mn + n + i \\ &= (n+1)i + n^2 + \frac{n(n+1)m}{2} \end{aligned}$$

$i = 1, 2, ..., m$ and $j = 1, 2 \ldots, n$.
All the vertex weights $v_i's$ are distinct.

Vertex weight of u_j,

$$\begin{aligned}
wt_f(u_1) &= f(u_1) + \sum_{i=1}^{m} f(u_1 v_i) \\
&= t + 1 + m + n + 1 + ..., m + n + m \\
&= t + 1 + m^2 + nm + \frac{m(m+1)}{2} \\
wt_f(u_2) &= f(u_2) + \sum_{i=1}^{m} f(u_2 v_i) \\
&= t + 2 + 2m^2 + nm + \frac{m(m+1)}{2}
\end{aligned}$$

$$wt_f(u_j) = t + j + jm^2 + nm + \frac{m(m+1)}{2}$$

All the vertex weights of $u_j's$ are distinct, and all the vertex weights of v_i and u_j are pairwise distinct.

Now the edge weights of $v_i u_j$ are

$$\begin{aligned}
wt_f(v_i u_j) &= f(v_i) + f(u_j) + f(v_i u_j) \\
&= i + t + j + jm + n + i \\
&= 2i + t + jm + n + j
\end{aligned}$$

For $i = 1, 2, ..., m$ and $j = 1, 2 \ldots, n$, All the edge weights of $v_i u_j$ are distinct. Now, the minimum vertex weight

$$\begin{aligned}
wt_f(u_1) &= t + 1 + m^2 + nm + \frac{m(m+1)}{2} \\
&= m + 1 + m^2 + nm + \frac{m(m+1)}{2}
\end{aligned}$$

Maximum edge weight

$$\begin{aligned}
wt_f(v_m u_n) &= 2m + m + nm + n + n \\
&= 3m + 2n + nm
\end{aligned}$$

The minimum vertex weight is greater than the maximum edge weight.
Case (ii) $m = n = t > 3$ is odd and if $m \neq n$
$1, 3, ..., 2t - 1$ for the vertices v_i and $2, 4, ..., 2t$ to the vertices u_j for each

$i = 1, 2, ..., m$ and $j = 1, 2, ..., n$ if $m = n = odd$.
$1, 3, ..., 2n-1, 2n+1, ..., n+m$ for the vertices v_i and $2, 4, ..., 2n$ to the vertices u_j for each $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$ if $m \neq n$.
Add the labels of the vertex to obtain the first edge $2t + 1$. Determine the sum of the vertex labels incident with the edge. For the successive $v_i u_1$ for each $i = 2, 3, ..., m$, Assign the consecutive integers in ascending order if the same sum of vertex labels to the edges. The complete bipartite graph $K_{5,5}$ as shown in figure 3. ☐

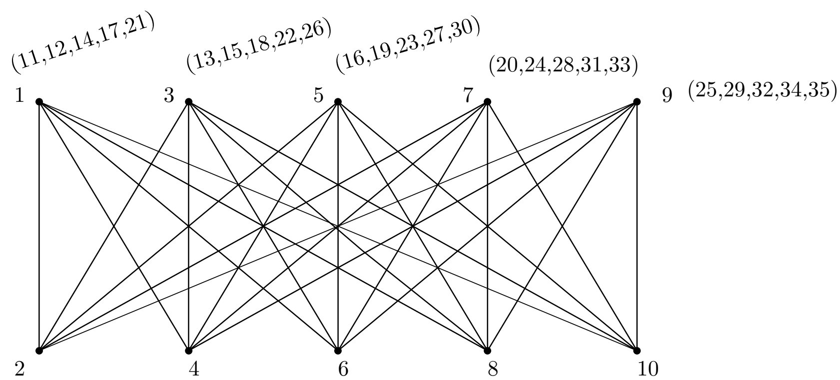

Figure 3: SVPTAT in $K_{5,5}$

3 Coexistence of vertex-magic and edge-antimagic on Comprehensive Graph Labeling

This section demonstrates several conditions under which a graph may accept or reject the simultaneous vertex-magic and edge-antimagic total labelings, as well as some graphs that admit both vertex-magic and edge-antimagic.

Theorem 3.1. *For every vertex in G, there may be a maximum of one pendant edge incident to it if the graph is both vertex-magic total and edge-antimagic total.*

Proof. Let G be a vertex-magic and edge-antimagic total graph at a time, and let its corresponding labeling be g. Assume that the vertex $u \in V(G)$ is adjacent to two vertex x and y of degree one. Since g is edge-antimagic total labeling, then

$$\begin{aligned} wt_g(ux) &\neq wt_g(uy) \\ g(u)+g(x)+g(ux) &\neq g(u)+g(y)+g(uy) \ \ implies \\ g(x)+g(ux) &\neq g(y)+g(uy) \\ i.e, wt_g(x) &\neq wt_g(y) \end{aligned}$$

which contradicts that g is vertex-magic total labeling. □

Theorem 3.2. *For every odd integer $n > 3$, $n \not\equiv 0(mod\ 3)$, the path graph P_n is vertex-magic total and edge-antimagic total.*

Proof. Define the path graph P_n's total labeling g, for odd integer $n > 3$, $n \not\equiv 0(mod\ 3)$ by

$$\begin{aligned} g(v_1) &= 2n-1 \\ g(v_i) &= 2i-3, \ for \ \ 2 \le i \le n \\ g(v_iv_{i+1}) &= \begin{cases} n-i, \ i \ is \ odd \\ 2n-i, \ i \ is \ even \end{cases} \end{aligned}$$

Then the vertex weights are $wt_g(v_i) = 3n-2$

With the weights mentioned above, it is clear that g is vertex-magic. Now, the edge weights are

$$wt_g(v_1v_2) = 3n-1$$

$$wt_g(v_iv_{i+1}) = \begin{cases} 3i+n-4, \ for \ i \ is \ odd \\ 3i+2n-4, \ for \ i \ is \ even \end{cases}$$

g is edge-antimagic with the weights given above. □

Note 3.3. *P_3 doesn't admit vertex magic and edge anti-magic.*

Suppose P_3 admits vertex magic and edge anti-magic labeling ψ. Then for any vertices v_1 and v_3 such that have the following contradictive conditions in P_3.

$$\begin{aligned} wt_\psi(v_1) &= wt_\psi(v_3) \\ \psi(v_1)+\psi(v_1v_2) &= \psi(v_3)+\psi(v_2v_3) \\ wt_\psi(v_1v_2) &\neq wt_\psi\psi(v_2v_3) \\ \psi(v_1)+\psi(v_2)+\psi(v_1v_2) &\neq \psi(v_2)+\psi(v_3)+\psi(v_2v_3) \\ \psi(v_1)+\psi(v_1v_2) &\neq \psi(v_3)+\psi(v_2v_3) \end{aligned}$$

Theorem 3.4. *The cycle graph C_n is vertex-magic and edge-antimagic total, for every odd integer $n \geq 3$, $n \not\equiv 0(mod\, 3)$.*

Proof. Let us consider a cycle graph C_n, where n is an odd integer $n \geq 3$, $n \not\equiv 0(mod\, 3)$. We define a total labeling g on C_n by

$$\begin{aligned} g(v_i) &= i,\ for\ 1 \leq i \leq n \\ g(v_i v_{i+1}) &= \begin{cases} \frac{1}{2}[4n+1-i],\ i\ is\ odd \\ \frac{1}{2}[3n+1-i],\ i\ is\ even \end{cases} \end{aligned}$$

It is evident that, with the aforementioned labeling g, C_n allows for both vertex-magic and edge-antimagic totals. □

Theorem 3.5. *If a graph G consists of a pair of adjacent vertices that are not adjacent to any other vertices except for a common vertex, then G does not admit both a vertex-magic and an edge-antimagic total labeling.*

Proof. Assume that g is a vertex-magic and edge-antimagic total labeling of a graph G. Let u and v be the pair of adjacent vertices that are only adjacent by w. Then

$$\begin{aligned} wt_g(u) &= wt_g(v) \\ g(u) + g(uv) + g(uw) &= g(v) + g(uv) + g(vw) \\ g(u) + g(uw) &= g(v) + g(vw) \end{aligned}$$

and also

$$\begin{aligned} wt_g(uw) &\neq wt_g(vw) \\ g(u) + g(uw) + g(w) &\neq g(v) + g(vw) + g(w) \\ g(u) + g(uw) &\neq g(v) + g(vw) \end{aligned}$$

which is contradiction to the fact (3). □

According to the abovementioned theorem, a graph G does not support both vertex-magic and edge-antimagic total labeling if at least one vertex in the graph is adjacent to both vertices of K_2. The following consequence is evident from the preceding theorem.

Corollary 3.6. *The friendship graph does not admit both vertex-magic and edge-antimagic total labeling.*

3.1 Corona product

The corona of graph G_1 with graph G_2, denoted as $G_1 \odot G_2$, is formed by taking one instance of G_1 and m instances of G_2, then connecting the j^{th} vertex of G_1 with an edge to each vertex in the j^{th} copy of G_2, where G_1 has order m.

Theorem 3.7. *For any graph G, the graph $G \odot mK_1$, for $m \geq 2$, does not admit both vertex-magic total and edge-antimagic total labeling.*

Proof. For example$m \geq 2$, the corona product of a graph mK_1 is a graph with m pendent edges at each vertex. According to lemma 3.1, $G \odot mK_1$ does not accept both vertex magic and edge antimagic for all $m \geq 2$. □

Theorem 3.8. *The graph $G \odot K_2$ does not admit both vertex-magic total and edge-antimagic total labeling for any graph G.*

Proof. The corona product of any graph G with K_2 is a graph whose vertex is incident with K_3. This refers to the order of G's pair of adjacent vertices, which, aside from a common vertex (the vertices of G), are not adjacent to any other vertices. According to theorem 3.5, it is not possible for $G \odot K_2$ to have both a vertex-magic and an edge-antimagic total labeling. □

Theorem 3.9. *For every $m \geq 1$, the graph $G \odot mK_2$ does not admit both vertex-magic total and edge-antimagic total labeling for any graph G.*

The proof of the theorem follows from theorem 3.8.

A graph known as a $m-$crown is constructed by connecting each vertex of a cycle graph with s pendant edges. This can be represented as $C_m \odot sK_1$, where $m = |V(C_n)|$. A sun graph is $1-$crown graph $(C_m \odot K_1)$.

In [7], Irfan and Semaničová-Feňovčíková mentioned the following open problem.

Open problem For the sun graph $Sun(n)$, $n \geq 3$, determine if there exists a total labeling that is simultaneously vertex-magic total and edge-antimagic total.

The following statement proves that the sun graph admits these labels. For example$m \geq 3$, the sun graph $C_m \odot K_1$ is both vertex-magic and edge-antimagic total.

Define a total labeling g from $V(C_n \odot K_1) \cup E(C_n \odot K_1)$ to $\{1, 2, \ldots, 4n\}$

in the following way:

$$\begin{aligned} g(x_i) &= 2i, \quad for \;\; 1 \leq i \leq n \\ g(y_1) &= 2n+2 \\ g(y_i) &= 4n+4-2i, \quad for \;\; 2 \leq i \leq n \\ g(x_1y_1) &= 4n-1 \\ g(x_iy_i) &= 2n-3+2i, \quad for \;\; 2 \leq i \leq n \\ g(x_ix_{i+1}) &= 2n+1-2i, \quad for \;\; 1 \leq i \leq n-1 \\ g(x_nx_1) &= 1. \end{aligned}$$

The vertex weights are then

$$wt_g(x_1) = wt_g(x_i) = wt_g(x_n) = wt_g(y_1) = wt_g(y_i) = 6n+1$$

As a result, the sun graph $C_n \odot K_1$, where $n \geq 3$, has a vertex-magic labeling for the total labeling, g. After that, the edge weights are

$$\begin{aligned} wt_g(x_1y_1) &= 6n+3 \\ wt_g(x_iy_i) &= 6n+2i+1, \quad for \;\; 2 \leq i \leq n \\ wt_g(x_ix_{i+1}) &= 2n+2i+3, \quad for \;\; 1 \leq i \leq n-1 \\ wt_g(x_nx_1) &= 2n+3 \end{aligned}$$

i.e, The above weights give us

$$\begin{aligned} & wt_g(x_nx_1) < wt_g(x_1x_2) < wt_g(x_2x_3) < \cdots < wt_g(x_{n-1}x_n) \\ < \;\; & wt_g(x_1y_1) < wt_g(x_2y_2) < \cdots < wt_g(x_ny_n) \end{aligned}$$

Every edge weight is obviously pairwise distinct based on the inequality above.

4 Conclusion

In this paper, we gave vertex antimagic constants and edge antimagic weights under the proposed labeling framework. This work contributes to the ongoing study of antimagic graph labeling and offers new insights into graph structures. We exhibited both vertex-magic total and edge-antimagic total

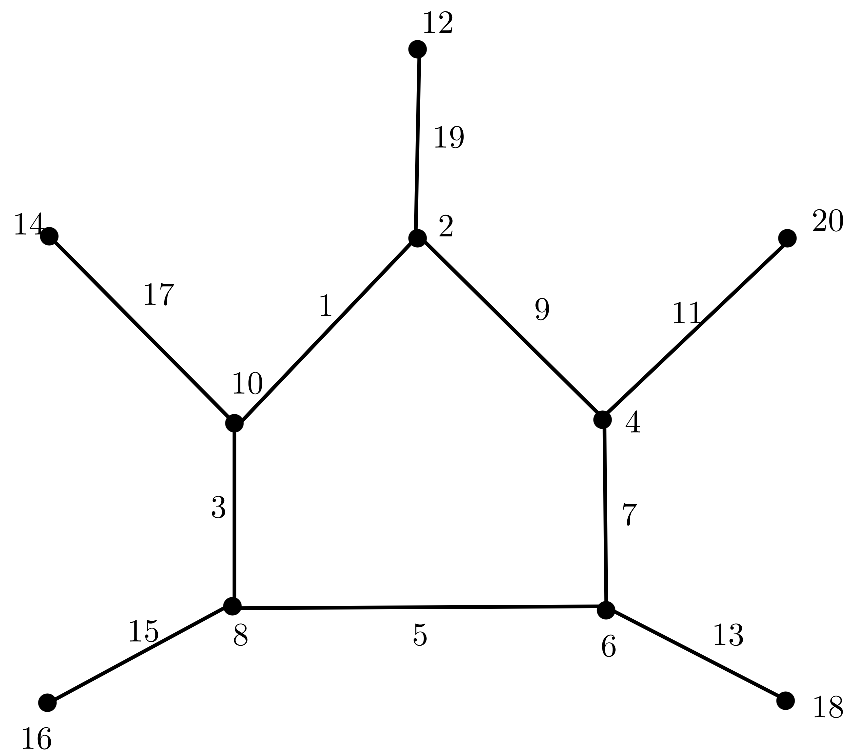

Figure 4: $C_5 \odot K_1$

properties simultaneously. Moreover, we presented criteria for a graph lacking vertex-magic total and edge-antimagic total labeling. Furthermore, we provided a comprehensive labeling of the sun graph that exhibits both vertex-magic and edge-antimagic properties, addressing an open problem presented by Irfan et al. [7] in 2016. We conclude our paper by suggesting the following open problems for consideration in future research.

Open problem 1: For a complete graph K_n, $n \geq 3$, determine the coexistence of vertex-magic and edge-antimagic total labeling.

Open Problem 2: For every even integer $n \geq 3$, $n \not\equiv 0(mod\ 3)$, find the path graph P_n and the cycle C_n that are admitted or not vertex-magic total and edge-antimagic total.

References

[1] G. Ali, M. Bača, M. Lascsáková, A. Semaničová-Feňovčíková, A. ALoqaily, N. Mlaiki, *Modular total vertex irregularity strength of graphs*,

AIMS Mathematics, **8,** no. 4, (2023), 7662–7671.

[2] M. Bača, F. Bertault, J.A. MacDougall, M. Miller, R. Simanjuntak, Slamin, *Vertex-antimagic total labelings of graphs*, Discussiones Mathematicae, **23,** (2003), 67–83.

[3] M. Bača, M. Miller, O. Phanalasy, J. Ryan, A. Semaničová-Feňovčíková, A. Abildgaard Sillasen, *Total labelings of graphs with prescribed weights*, J. Combin. Math. Combin. Comput., **120,** (2024), 367–376.

[4] M. Bača, M. Miller, O. Phanalasy, A. Semaničová-Feňovčíková, A. Abildgaard Sillasen, *Totally antimagic total graphs*, Australian J. Combin., **61**, no. 1, (2015), 42–56.

[5] S. Balasundar, R. Jeyabalan, R. Nishanthini, P. Swathi, *Strongly vertex perfectly antimagic total graphs*, IAENG International Journal of Applied Mathematics, **54**, no. 8, (2024), 1597–1601.

[6] J.A. Gallian, *A dynamic survey of graph labeling*, Electron. J. Combin., **16,** (2013) #DS6.

[7] Muhammad Irfan, A. Semaničová-Feňovčíková, *On total labelings of graphs with prescribed weights*, AKCE International J. Graphs and Combin., **13,** (2016), 191–199.

[8] R. Simanjuntak, F. Bertault, M. Miller, *Two new (a, d)-antimagic graph labelings*, Proc. Eleventh Australasian Workshop On Combinatorial Algorithms, **11,** (2000), 179–189.

[9] Siti' Aisyah Nur Ni'mah, Diari Indriati, *On vertex irregular total k-labeling and total vertex irregularity strength of lollipop graphs*, J. Physics: Conference Series, 1306, (2019).

[10] K.A. Sugeng, M. Miller, *On consecutive edge magic total labeling of graphs*, Journal of Discrete Algorithms, **6,** (2008), 59–65.

[11] P. Swathi, G. Kumar, R. Jeyabalan, R. Nishanthini, *Perfectly Antimagic Total Graphs*, Journal of Intelligent and Fuzzy Systems, **44,** no.1, (2023), 1517–1523.

[12] P. Swathi, R. Jeyabalan, *Perfectly antimagic total labeling of star-like trees and the corona product of two graphs*, Acta Mathematica Universitatis Comenianae, **93,** no. 2, (2024), 95–100.