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An Application of Hypergeometric Functions in the Evaluation of a Gradshteyn-Ryzhik Integral

Juan Carlos López Carreño[1], **Rosalba Mendoza Suárez**[1], **Jairo Alonso Mendoza S.**[2]

[1]Department of Mathematics
Faculty of Basic Sciences
University of Pamplona
Pamplona, Colombia

[2]Department of Physics
Faculty of Basic Sciences
University of Pamplona
Pamplona, Colombia

email: jairoam@unipamplona.edu.co

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Abstract

In this article, we evaluate an interesting integral from the famous book of integrals by Gradshteyn and Ryzhik [5], which, in its seventh edition, includes an incorrect result. We derive the correct value. Moreover, we highlight the application of the hypergeometric functions formalism in the evaluation of this integral.

1 Introduction

The tables of series and integrals have been used over time. Among these, we can mention [1], [2], [3] [4]. After a search, we found that the table of

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integrals by Gradshteyn and Ryzhik (see [5]) is the most popular among users of the scientific community. On page 531, section integral 4.224.13 of [5], the following integral appears:

$$\int_0^{\frac{\pi}{2}} \ln(1+2a\sin x+a^2)dx = \sum_{k=0}^{\infty} \frac{2^{2k}(k!)^2}{(2k+1)(2k+1)!!}\Big(\frac{2a}{1+a^2}\Big)^{2k+1}, \ a^2 \leq 1 \tag{1.1}$$

there is an error in the previous formula, as it can be observed that the series is not convergent.

The objective of this article is to derive a correct formula that expresses the integral in (1.1) as a convergent series.

2 Evaluating the integral (1.1)

Theorem 2.1. *If $a^2 \leq 1$, the following integral formula holds:*

$$\int_0^{\frac{\pi}{2}} \ln(1+2a\sin x+a^2)dx = \sum_{k=0}^{\infty} \frac{2^k k!}{(2k+1)\cdot(2k+1)!!} \cdot \Big(\frac{2a}{1+a^2}\Big)^{2k+1}.$$

Demostration 2.1. *Let*

$$I(a) = \int_0^{\frac{\pi}{2}} \ln(1+2a\sin x+a^2)dx.$$

The expression inside the logarithm of the previous integral can be written as a product of simpler terms. To achieve this, we note that

$$1+2a\sin x+a^2 = (1+a^2)\Big(1+\frac{2a}{1+a^2}\sin x\Big)$$

where from,

$$I(a) = \int_0^{\frac{\pi}{2}} \ln\Big[(1+a^2)\Big(1+\frac{2a}{1+a^2}\sin x\Big)\Big]dx,$$

we use the logarithm property to split the integral into two parts as follows:

$$I(a) = \int_0^{\frac{\pi}{2}} \ln(1+a^2)dx + \int_0^{\frac{\pi}{2}} \ln(1+b\sin x)dx,$$

where

$$b = \frac{2a}{1+a^2}.$$

If we denote the second integral on the right-hand side of the previous equality by $F(b)$, we have that:

$$I(a) = \frac{\pi}{2}\ln(1+a^2) + F(b). \tag{2.2}$$

To evaluate the integral $F(b)$, we use the power series expansion of $\log(1+z)$:

$$\log(1+z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{n+1}, \ |z| < 1,$$

with $z = b\sin x$ and so

$$F(b) = \int_0^{\frac{\pi}{2}} \ln(1+z)dz = \int_0^{\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n b^{n+1}\sin^{n+1} x}{n+1} dx.$$

Now, the uniform convergence of the above power series allows us to interchange the integral and the summation as follows:

$$F(b) = \sum_{n=0}^{\infty} \frac{(-1)^n b^{n+1}}{n+1} \int_0^{\frac{\pi}{2}} \sin^{n+1} x dx,$$

an expansion that we can rewrite in the form:

$$F(b) = \sum_{k=0}^{\infty} \frac{b^{2k+1}}{2k+1} \int_0^{\frac{\pi}{2}} \sin^{2k+1} x dx - \sum_{k=0}^{\infty} \frac{b^{2k+2}}{2k+2} \int_0^{\frac{\pi}{2}} \sin^{2k+2} x dx. \tag{2.3}$$

Using the well-known Wallis formulas [6] :

$$\int_0^{\frac{\pi}{2}} \sin^{2k+1} x dx = \frac{2^k k!}{(2k+1)!!}$$

and

$$\int_0^{\frac{\pi}{2}} \sin^{2k} x dx = \frac{(2^k+1)!!\pi}{2^{k+1}(2k+1)k!}.$$

The equality (2.3) can be written as

$$F(b) = \sum_{k=0}^{\infty} \frac{2^k k!\, b^{2k+1}}{(2k+1)\cdot(2k+1)!!} - \frac{\pi b^2}{4} \sum_{k=0}^{\infty} \frac{(2k+3)!!\, b^{2k}}{2^{k+1}(k+1)(k+1)!(2k+3)}. \tag{2.4}$$

Therefore, from (2.2), (2.3), and (2.4), it follows that:

$$I(a) = \sum_{k=0}^{\infty} \frac{2^k k!}{(2k+1)\cdot(2k+1)!!} \cdot \left(\frac{2a}{1+a^2}\right)^{2k+1}.$$

Proposition 2.2. *For* $b = \frac{2a}{1+a^2}$*, we have*

$$G(b) := -\frac{\pi b^2}{4} \sum_{k=0}^{\infty} \frac{(2k+3)!!\, b^{2k}}{2^{k+1}(k+1)(k+1)!(2k+3)} = \frac{\pi}{2} \log(1+a^2). \tag{2.5}$$

Demostration 2.2. *Recall that the Pochhammer symbol* $(\alpha)_n$*, defined as:*

$$\begin{aligned} &(\alpha)_n := \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1), \quad n \in \mathbb{N}, \\ &(\alpha)_0 := 1, \end{aligned}$$

satisfies the following properties:

$$\begin{aligned} &(\alpha)_{n+1} = \alpha(\alpha+1)_n \\ &(1)_n = n! \\ &(\alpha)_{2n} = 2^{2n}\left(\frac{\alpha}{2}\right)_n \left(\frac{\alpha+1}{2}\right)_n, \end{aligned}$$

the coefficient in the power series shown in (2.5) can be written as:

$$\frac{(2k+3)!!}{2^{k+1}(k+1)(k+1)!(2k+3)} = \frac{1\cdot 3\cdots(2k+1)}{2^{k+1}(k+1)(1)_{k+1}} = \frac{1\cdot 2\cdot 3\cdot 4\cdot\cdot(2k)(2k+1)}{2^{k+1}(k+1)(1)_{k+1}2\cdot 4\cdot\cdot(2k)}$$

$$= \frac{(2k+1)!}{2^{2k+1}(k+1)!(1)_{k+1}} = \frac{(1)_{2k+1}}{2^{2k+1}(1)_{k+1}(1)_{k+1}}$$

$$= \frac{(2)_{2k}}{2^{2k+1}(2)_k(2)_k} = \frac{(1)_k\left(\frac{3}{2}\right)_k(1)_k}{2(2)_k(2)_k} \cdot \frac{1}{k!}.$$

Therefore, the series in (2.5) can be written as:

$$G(b) := -\frac{\pi b^2}{8} \sum_{k=0}^{\infty} \frac{(1)_k(1)_k\left(\frac{3}{2}\right)_k}{(2)_k(2)_k} \cdot \frac{(b^2)^k}{k!}.$$

The previous expression corresponds to a hypergeometric function as follows:

$$G(b) = -\frac{\pi b^2}{8} \; {}_3F_2\left(\left. \begin{matrix} 1, 1, \frac{3}{2} \\ 2, 2 \end{matrix} \right| b^2 \right). \tag{2.6}$$

The hypergeometric function that appears in (2.6) can be evaluated exactly (see https://functions.wolfram.com/HypergeometricFunctions/Hypergeometric 3F2/03/08/06/01/02/08/0001/).

$$ {}_3F_2\left(\begin{matrix} 1,1,\frac{3}{2} \\ 2,2 \end{matrix} \middle| z \right) = -\frac{4}{z}\log\left(\frac{1+\sqrt{1-z}}{2}\right). \tag{2.7} $$

Taking $z = b^2$ in (2.7), it follows that:

$$ {}_3F_2\left(\begin{matrix} 1,1,\frac{3}{2} \\ 2,2 \end{matrix} \middle| b^2 \right) = -\frac{4}{b^2}\log(1+a^2), $$

and so, from this equality and (2.6), we have:

$$ G(b) = \frac{\pi}{2}\log(1+a^2). $$

Consequently, proposition 2.2 follows.

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