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Improving the approximation order of Baskakov-Beta operators based on a parameter

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Abstract

In this paper, we construct a new Baskakov-Beta-type operator based on two sequences. We discuss uniform convergence and study various approximation properties. We find a recurrence relation to calculate the moments for this operator. We introduce a modified Baskakov-Beta operators based on parameter S. We give some direct estimates for this operator using the first and second modulus of continuity.

1 Introduction

The well-known classical Baskakov sequence [1] is defined as

$$V_n(f;x) = \sum_{k=0}^{\infty} q_{n,k}(x) f\left(\frac{k}{n}\right), \quad q_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}, x \in [0,\infty). \tag{1.1}$$

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Several researchers have studied various modifications of the Baskakov-Durrmeyer operators [2, 3, 4]. In a recent paper, Jabbar and Hassan [5] have introduced a sequence of modified Baskakov operators as follows:

$$B_n^1(f;x) = \sum_{k=0}^{\infty} q_{n,k}^1(x) f\left(\frac{k}{n}\right) \tag{1.2}$$

$q_{n,k}^1(x) = \psi(x,n)q_{n+1,k}(x) + \psi(-1-x,n)q_{n+1,k-1}(x)$, $\psi(x,n) = a(n) + b(n)x$, where $a(n)$ and $b(n)$ are unknown sequences. Obviously, for $a(n) = b(n) = 1$, we get (1.1).

2 Beta Operators of Order 1

Let $\gamma > 0$ and let $C_\gamma(I) = \{f \in C(I) : |f(x)| \leq M(1 + x^\gamma), \forall x \in I\}$, where $I = [0, \infty)$ and M is positive constant dependent on f. For any $f \in C_\gamma(I)$, we define a new Baskakov-Durrmeyer operator as

$$D_{n,k}^1(f;x) = \sum_{k=0}^{\infty} q_{n,k}^1(x) \int_0^\infty B_{n,k}(t) f(t)dt, x \in I, \tag{2.3}$$

where $B_{n,k}(t) = \frac{(n+k)!t^k}{k!(n-1)!(1+t)^{n+k+1}}$.

Lemma 2.1. *The following statements hold*

i) $D_{n,k}^1(1;x) = 2a(n) - b(n)$.

ii) $D_{n,k}^1(t;x) = (2a(n) - b(n))x + \frac{(4a(n)-3b(n))x+(3a(n)-2b(n))}{n-1}$.

iii) $D_{n,k}^1(t^2;x) = (2a(n)-b(n))x^2 + n\frac{\left[\left(12x^2+10x\right)a(n)-\left(8x^2+6x\right)b(n)\right]}{(n-1)(n-2)} + \frac{\left[(8+10x)a(n)-\left(6+10x+2x^2\right)b(n)\right]}{(n-1)(n-2)}$.

$$2a(n) - b(n) = 1 \tag{2.4}$$

We will consider two cases for the sequences $a(n)$ and $b(n)$.
Case 1: *Let*

$$a(n) - b(n) \geq 0 \text{ and } a(n) \geq 0. \tag{2.5}$$

Using (2.4), we get $0 \leq a(n) \leq 1$ and $-1 \leq b(n) \leq 1$. In this case, the operator (2.4) is positive.
Case 2: *Let*

$$a(n) - b(n) < 0 \text{ or } a(n) < 0. \tag{2.6}$$

If $a(n) - b(n) < 0$, then $a(n) > 1$ and if $a(n) < 0$, then $a(n) - b(n) > 1$. In this case, the operator (2.3) is non-positive.

Theorem 2.2. *Let $a(n)$ and $b(n)$ be two sequence satisfying the conditions (2.4) and (2.5). If $f \in C_\gamma(I)$, then $\lim_{n\to\infty} D_n^1(f;x) = f(x)$ uniformly on $[u,v] \subset [0,\infty)$.*

Proof. In view of Lemma 2.1 and Korovkin theorem, the result of this theorem follows easily. □

Theorem 2.3. *The m-th order moment for the operators (2.3) for $m \in N^0$ is defined as*

$$T_{n,m}(x) = D_n^1\left((t-x)^m;x\right) = \sum_{k=0}^{\infty} q_{n,k}^1(x) \int_0^\infty B_{n,k}(t)(t-x)^m dt \qquad (2.7)$$

Then the recurrence relation is given by

$$(n-m-2)T_{n,m+1}(x) = x(1+x)\left(T'_{n,m}(x) + 2mT_{n,m-1}(x)\right) + (1+2x)(m+1)T_{n,m}(x) + (a(n)-b(n))\left(x\lambda_{n,m}(x) + (1+x)\mu_{n,m}(x)\right), \qquad (2.8)$$

where

$$\lambda_{n,m}(x) = \sum_{k=0}^{\infty} q_{n+1,k}(x) \int_0^\infty B_{n,k}(t)(t-x)^m dt \quad \mu_{n,m}(x) = \sum_{k=0}^{\infty} q_{n+1,k-1}(x) \int_0^\infty B_{n,k}(t)(t-x)^m dt.$$

For each $x \in [0,\infty), T_{n,m}(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right)$ where $[\,\delta]$ denotes the integer part of δ.

Theorem 2.4. *Let $a(n)$ and $b(n)$ be convergent sequences that satisfy the conditions (2.4) and (2.5). If $f'' \in C_\gamma(I)$, then*

$$\lim_{n\to\infty} n\left(D_{n,k}^1(f;x) - f(x)\right) = \left((4L_1 - 3L_2)x + (3L_1 - 2L_2)\right) f'(x) + x(1+x)f''(x),$$

where $\lim_{n\to\infty} a(n) = L_1$ and $\lim_{n\to\infty} b(n) = L_2$.

Proof. The result follows from Taylor's formula and theorem. □

3 Beta Operators of Order of approximation $O\left(\frac{1}{n^{S+1}}\right)$

In this section, we extend the previous results considering a new Durrmeyer operator that has order of approximation $O\left(\frac{1}{n^{S+1}}\right)$ as follows:

$$D_n^{\mathrm{S}}(f;x) = \sum_{k=0}^{\infty} q_{n,k}^{1}(x) \int_0^{\infty} B_{n,k}(t) f\left(x + \frac{(t-x)}{n^S}\right) dt \tag{3.9}$$

If $S = 0$, then the operator (3.9) reduces to the operator (2.3).

Lemma 3.1. *By simple computation, we have*

i) $D_n^{\mathrm{S}}(1;x) = 2a(n) - b(n)$

ii) $D_n^{\mathrm{S}}(t;x) = (2a(n) - b(n))x + \frac{(4a(n)-3b(n))x+(3a(n)-2b(n))}{n^s(n-1)}$

iii) $D_n^S(t^2;x) = (2a(n) - b(n))x^2$

$$+ \frac{[(8x^2+6x)\,a(n) - (6x^2+4x)\,b(n)]}{n^{s-1}(n-1)(n-2)} + \frac{[(4x^2+4x)\,a(n) - (2x^2+2x)\,b(n)]}{n^{2s-1}(n-1)(n-2)}$$
$$- \frac{[(12x+16x^2)\,a(n) - (8x+12x^2)\,b(n)]}{n^s(n-1)(n-2)}$$
$$+ \frac{[(8+22x+16x^2)\,a(n) - (6+18x+14x^2)\,b(n)]}{n^{2s}(n-1)(n-2)}$$

Theorem 3.2. *Let* $f \in C_\gamma(I)$ *and let the operator* (3.9) *be non-positive. We have* $\lim_{n\to\infty} D_n^S(f;x) = f(x)$ *uniformly on* $[u,v] \subset [0,\infty)$*.*

Proof. The Operator $D_n^S(f;x)$ can be written as follows:

$$D_n^S(f;x) = U_n^S(f;x) - W_n^S(f;x),$$

where

$$U_n^S(f;x) = \sum_{k=0}^{\infty} ((2a(n) + b(n)x)q_{n+1,k}(x) + a(n)q_{n+1,k-1}) \int_0^{\infty} B_{n,k}(t)dt$$
$$W_n^S(f;x) = \sum_{k=0}^{\infty} (a(n)q_{n+1,k}(x) + (b(n)x + b(n))q_{n+1,k-1}) \int_0^{\infty} B_{n,k}(t)dt.$$

From (2.4) and (2.7), it follows that $(a(n) > 0, b(n) > 0)$ or $(a(n) < 0, b(n) < 0)$.
Suppose that $a(n) > 0, b(n) > 0$. By using Theorem 10 in [6], we obtain

$$\lim_{n\to\infty} U_n^S(f;x) = \left(L_2\left(x+\frac{3}{2}\right)+\frac{3}{2}\right)f(x)$$
$$\lim_{n\to\infty} W_n^S(f;x) = \left(L_2\left(x+\frac{3}{2}\right)+\frac{1}{2}\right)f(x).$$

Hence, $\lim_{n\to\infty} D_n^S(f;x) = f(x)$. □

Theorem 3.3. *Let the operator* (3.9) *be positive operator and let* $f'' \in C_\gamma(I)$. *Then*

$$\lim_{n\to\infty} n^{s+1}\left(D_n^{\mathrm{S}}(f;x)-f(x)\right) = \left(\left(4L_1-3L_2\right)x+\left(3L_1-2L_2\right)\right)f'(x) \quad (3.10)$$

Proof. By Applying Taylor's formula on the function f, we obtain

$$f(t) = f(x)+(t-x)f'(x)+\frac{1}{2}(t-x)^2f''(x)+\theta(t,x)(t-x)^2,$$

where $\theta(t,x) \to 0$ as t $\to x$ and $\theta \in C_\gamma(I)$.

Hence,

$$\lim_{n\to\infty} n^{S+1}\left(D_n^S(f;x)-f(x)\right) = (1+2x)\left(3L_1-2L_2\right)f'(x) + \lim_{n\to\infty} n^{S+1}D_{n,k}^S\left(\theta(t,x)(t-x)^2;x\right)$$

Using the Cauchy-Schwarz inequality for the positive operators $D_{n,k}^S$, we get

$$\lim_{n\to\infty} n^{s+1}D_n^S\left(\theta(t,x)(t-x)^2;x\right) = 0.$$

The proof of this theorem is complete. □

For $\delta > 0$, the K-functional is given by

$$K(f;\delta) = \inf\left\{\|f-g\|+\delta\|g''\|_{C(I_h)}, g \in C_\gamma^2(I)\right\}, \quad (3.11)$$

where $C_\gamma^2(I) = \{g \in C(I_h) : g', g'' \in C_\gamma(I)\}$. There exists a constant $M > 0$ such that

$$K(f;\delta) \leq M\omega_2(f;\sqrt{\delta}). \tag{3.12}$$

where $\omega_2(f;\sqrt{\delta})$ is the Second order Modulus of continuity.

Theorem 3.4. *For $f \in C_\gamma(I)$, we have the following inequality:*

$$\left|D_n^{\mathrm{S}}(f;x) - f(x)\right| \leq M\omega_2\left(f;\frac{1}{2}\sqrt{T_{n,2}^{\mathrm{S}}(x) + \left(T_{n,1}^{\mathrm{S}}(x)\right)^2}\right) + \omega\left(f;T_{n,1}^{\mathrm{S}}(x)\right),$$

where $T_{n,1}^{\mathrm{S}}(x) = D_n^{\mathrm{S}}(t-x;x)$ and $T_{n,2}^{\mathrm{S}}(x) = D_n^{\mathrm{S}}\left((t-x)^2;x\right)$.

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