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Another Look at a Definite Integral

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Abstract

In this article, we revisit a specific integral previously analyzed using complex analysis and branch points. A generalized form of this integral is derived using Cauchy's residue theorem, followed by an additional generalization employing hypergeometric functions.

1 Introduction

The integral

$$\int_a^b (x^2-a^2)^{\frac{1}{2}}(b^2-x^2)^{\frac{1}{2}}\frac{dx}{x}, \tag{1.1}$$

was considered in [1], where the authors applied the residue theorem from complex analysis, for which they needed to introduce two branch points. In Section 2, we use Cauchy's residue theorem, to obtain the following generalization of (1.1):

$$\int_a^b (x^2-a^2)^{\frac{1}{2n}}(b^2-x^2)^{1-\frac{1}{2n}}\,dx = \frac{\pi b^2}{4}\left[\left(\frac{a}{b}\right)^2 + 2n\left(\frac{a}{b}\right)^{\frac{1}{n}} + 2n - 1\right]. \tag{1.2}$$

Then, in Section 3, we present a generalization of the above integrals using hypergeometric functions.

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2 Generalization of the integral (1.1)

For $n \in \mathbb{N}$, consider the integral J, defined for real numbers with $0 < a < b$ by

$$J = \int_a^b (x^2 - a^2)^{\frac{1}{2n}} (b^2 - x^2)^{1-\frac{1}{2n}} \, \frac{dx}{x}.$$

By the substitution

$$z^{2n} = \frac{x^2 - a^2}{b^2 - x^2},$$

the integral J can be written in terms of the new integration variable z as

$$J = n(b^2 - a^2)^{2n} \int_0^{\infty} \frac{z^{2n}}{(1 + z^{2n})^2 (a^2 + b^2 z^{2n})} \, dz.$$

Considering that z is in the integrand and in the parity function, with $c = \frac{a}{b}$, the integral J can be written as

$$J = \frac{n(b^2 - a^2)^{2n}}{2b^2} \int_{-\infty}^{\infty} \frac{z^{2n}}{(1 + z^{2n})^2 (c^2 + z^{2n})} \, dz. \qquad (2.3)$$

Let $z \in \mathbb{C}$ and consider the function $f(z)$ given by

$$f(z) = \frac{z^{2n}}{(1 + z^{2n})^2 (c^2 + z^{2n})}.$$

Observe that

$$z_k = c^{1/n} e^{\frac{(1+2k)\pi}{2n} i}, \qquad k = 0, 1, 2, \ldots, 2n - 1. \qquad (2.4)$$

are simple poles of f in the complex plane, of which for $k = 0, 1, 2, \ldots, n-1$, are located in the upper half-plane. Therefore, the residue of f at z_k [2, p. 132] is given by

$$\operatorname{Res}(f, z_k) = \lim_{z \to z_k} \frac{z^{2n}}{(1 + z^{2n})^2} \frac{(z - z_k)}{(c^2 + z^{2n})} = \frac{z_k^{2n}}{(1 + z_k^{2n})^2} \lim_{z \to z_k} \frac{(z - z_k)}{c^2 + z^{2n}}.$$

Using L'Hôpital's rule to evaluate the previous limit, we have:

$$\operatorname{Res}(f, z_k) = \frac{z_k^{2n}}{(1 + z_k^{2n})^2} \cdot \frac{1}{2n z_k^{2n-1}} = \frac{1}{2n} \frac{z_k}{(1 + z_k^{2n})^2}.$$

Taking into account the expression of z_k given in (2.3) and using Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$, the previous residue simplifies to:

$$\operatorname{Res}(f, z_k) = \frac{c^{\frac{1}{n}} e^{\frac{(1+2k)}{2n}\pi i}}{2n(1-c^2)^2}, \qquad k = 0, 1, 2, \ldots, n-1. \tag{2.5}$$

On the other hand, observe that w_k, given by $w_k = e^{\frac{(1+2k)\pi}{2n}i}$, corresponds to double poles of f in the upper half-plane.

Define $g(z) = \frac{z^{2n}}{c^2+z^{2n}}$. Then $g'(z) = \frac{2nc^2z^{2n}}{z(c^2+z^{2n})^2}$. Calculating these limits and derivatives [2, p. 132], it follows that

$$\operatorname{Res}(f, w_k) = \lim_{z\to w_k} \frac{d}{dz}\left[\left(g(z)\frac{z-w_k}{1+z^{2n}}\right)^2\right] = \frac{(1-2n-c^2)}{4n^2(c^2-1)^2} e^{\frac{(1+2k)\pi}{2n}i}. \tag{2.6}$$

The Cauchy Residue Theorem allows us to write

$$\int_{-\infty}^{\infty} \frac{z^{2n}}{(1+z^{2n})^2(c^2+z^{2n})}\, dz = 2\pi i\left[\sum_{k=0}^{n-1}\operatorname{Res}(f, z_k) + \sum_{k=0}^{n-1}\operatorname{Res}(f, w_k)\right].$$

Taking into account the relationships given in (2.5) and (2.6), the previous integral can be expressed as

$$\int_{-\infty}^{\infty} \frac{z^{2n}}{(1+z^{2n})^2(c^2+z^{2n})}\, dz = 2\pi i\frac{\left(1-c^2+2n\ c^{1/n}-2n\right)}{4n^2(1-c^2)^2}\sum_{k=0}^{n-1} e^{\frac{(1+2k)\pi}{2n}i}.$$

Using the geometric series to evaluate the previous sum, we have

$$\sum_{k=0}^{n-1} e^{\frac{(1+2k)\pi}{2n}i} = e^{\frac{\pi i}{2n}}\sum_{k=0}^{n-1} e^{\frac{k\pi}{n}i} = e^{\frac{\pi i}{2n}}\cdot\left[\frac{1-e^{\pi i}}{1-e^{\frac{\pi i}{n}}}\right] = 2\frac{e^{\frac{\pi i}{2n}}}{1-e^{\frac{\pi i}{n}}} = \frac{2}{e^{-\frac{\pi i}{2n}}-e^{\frac{\pi i}{2n}}} = \frac{i}{\sin\left(\frac{\pi}{2n}\right)}.$$

Substituting the value of this sum into the indefinite integral, it follows that

$$\int_{-\infty}^{\infty} \frac{z^{2n}}{(1+z^2)^2(c^2+z^{2n})}\, dz = \frac{\pi(c^2+2n-1-2nc^{1/n})}{2n^2(c^2-1)^2}\cdot\frac{1}{\sin(\frac{\pi}{2n})}.$$

Finally, from the above equality, equation (2.3), and $c = \frac{a}{b}$, it follows that

$$\int_a^b (x^2-a^2)^{\frac{1}{2n}}(b^2-x^2)^{1-\frac{1}{2n}}\, dx = \frac{\pi b^2}{4}\left[\left(\frac{a}{b}\right)^2 + 2n\left(\frac{a}{b}\right)^{\frac{1}{n}} + 2n - 1\right]. \tag{2.7}$$

Note 1: *If we set $n = 1$ in* (2.7)*, then we get the result of* (1.1) *considered in [1].*

3 Generalization of the integral (1.2)

Let us consider the defined integral I *given by*

$$I = \int_a^b (x^2 - a^2)^\alpha (b^2 - x^2)^{1-\alpha} \frac{dx}{x},$$

where $0 < \alpha < 1$ *and* $0 < a < b$*. Substituting* $x^2 - a^2 = (b^2 - a^2)t$*, the integral* I *can be expressed as*

$$I = \frac{(b^2 - a^2)^2}{2a^2} \int_0^1 t^\alpha (1-t)^{1-\alpha} \left[1 - \left(1 - \frac{1}{c^2}\right) t\right]^{-1} dt \quad \text{where} \quad c = \frac{a}{b}.$$

Using Euler's integral representation (see Askey, Theorem 2.2.1, p. 65)

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \middle| x\right),$$

with the identification $b = 1 + \alpha$*,* $c = 3$*,* $a = 1$*, and* $x = 1 - \frac{1}{c^2}$*, the integral* I *takes the form*

$$I = \frac{(b^2 - a^2)^2}{4a^2} \Gamma(1+\alpha)\Gamma(2-\alpha)\ {}_2F_1\left(\begin{matrix} 1 & 1+\alpha \\ & 3 \end{matrix} \middle| x\right). \tag{3.8}$$

In order to evaluate the hypergeometric function that appears on the right side of (3.8)*, we make use of the identity [3, formula 2.3.13, p. 79]*

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix} \middle| x\right) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1\left(\begin{matrix} a \quad b \\ a+b+1-c \end{matrix} \middle| 1-x\right) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-x)^{c-a-b} {}_2F_1\left(\begin{matrix} c-a \quad c-b \\ 1+c-a-b \end{matrix} \middle| 1-x\right) \end{aligned}$$

q with b, c, a, x *as defined earlier. Thus, from* (3.8)*, we obtain*

$$\begin{aligned} I &= \frac{(b^2 - c^2)^2}{2a^2} \Gamma(1+\alpha)\Gamma(1-\alpha)\ {}_2F_1\left(\begin{matrix} 1 \quad 1\ +\alpha \\ \alpha \end{matrix} \middle| \frac{1}{c^2}\right) \\ &+ \frac{(b^2 - c^2)^2}{2a^2} \frac{\Gamma(\alpha - 1)\Gamma(2-\alpha)}{(c^2)^{1-\alpha}}\ {}_2F_1\left(\begin{matrix} 2 \quad 2-\alpha \\ 2-\alpha \end{matrix} \middle| \frac{1}{c^2}\right). \end{aligned} \tag{3.9}$$

Using the formula that expresses the function $(1-z)^{-d}$ *as a hypergeometric series* $(1-z)^{-d} = {}_1F_0(\ d\ \mid z)$*, we have*

$${}_2F_1\left(\begin{matrix} 2 \quad 2-\alpha \\ 2-\alpha \end{matrix} \middle| \frac{1}{c^2}\right) = {}_1F_0\left(\ 2\ \middle| \frac{1}{c^2}\right) = \frac{c^4}{(c^2-1)^2}. \tag{3.10}$$

Applying the recurrence relation of the gamma function and Euler's reflection formula we obtain

$$\Gamma(1+\alpha)\Gamma(1-\alpha) = \frac{\alpha\pi}{\sin\alpha\pi}, \quad \textit{and} \quad \Gamma(\alpha-1)\Gamma(2-\alpha) = -\frac{\pi}{\sin\alpha\pi}. \tag{3.11}$$

Considering (3.10) *and* (3.11), *the expression given in* (3.9) *can be written as*

$$I = \frac{(b^2-a^2)^2}{2a^2}\frac{\alpha\pi}{\sin\alpha\pi}\left[{}_2F_1\left(\begin{array}{cc} 1 & 1+\alpha \\ & \alpha \end{array}\middle| \frac{1}{c^2}\right) - \frac{c^{2+2\alpha}}{\alpha(c^2-1)^2}\right]. \tag{3.12}$$

Finally, an application of the transformation:

$${}_2F_1\left(\begin{array}{cc} a & b \\ & b-1 \end{array}\middle| z\right) = \frac{(1-z)^{-a-1}\left[b-1+(a-b+1)z\right]}{b-1},$$

with $a=1,\ b=1+\alpha, z=\frac{1}{c^2}$, *allows us to write*

$${}_2F_1\left(\begin{array}{cc} 1 & 1+\alpha \\ & \alpha \end{array}\middle| \frac{1}{c^2}\right) = c^4\left[\frac{\alpha+(1-\alpha)(\frac{1}{c^2})}{\alpha(c^2-1)^2}\right], \tag{3.13}$$

recalling that $c=\frac{a}{b}$. *Substituting* (3.13) *into* (3.12) *yields the value of the integral* I

$$I = \frac{b^2}{2}\frac{\alpha\pi}{\sin(\alpha\pi)}\left[\left(\frac{1}{\alpha}-1\right)+\frac{a^2}{b^2}-\frac{1}{\alpha}\left(\frac{a}{b}\right)^{2\alpha}\right]. \tag{3.14}$$

Note 2. *In formula* (3.14), *if we take* $\alpha=\frac{1}{2}$, *then we obtain*

$$I = \frac{b^2\pi}{4}\left[(2-1)+\frac{a^2}{b^2}-2\left(\frac{a}{b}\right)\right] = \frac{\pi b^2}{4}\left(1-\frac{a}{b}\right)^2,$$

which corresponds to the value found in [1].

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