International Journal of Mathematics and Computer Science
Volume **20,** Issue no. 2, (2025), 641–649
DOI: https://doi.org/10.69793/ijmcs/02.2025/EdwinZarrazola

Product and quotient of independent Topp-Leone variables

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(Received January 22, 2025, Accepted March 19, 2025, Published March 20, 2025)

Abstract

The univariate family of Topp-Leone distributions has been studied recently by a number of researchers. This distribution has finite support and possesses many interesting properties. In this article, we derive the exact distributions of X_1X_2, X_1/X_2 and $X_1/(X_1+X_2)$ when X_1 and X_2 are independent Topp-Leone/beta variables. The resulting distributions are expressed in terms of special functions.

1 Introduction

A random variable X is said to have a Topp-Leone distribution, denoted by $X \sim \mathrm{TL}(\nu;\sigma)$, if its pdf is given by

$$f_{\mathrm{TL}}(x;\nu,\sigma) = \frac{2\nu}{\sigma}\left(\frac{x}{\sigma}\right)^{\nu-1}\left(1-\frac{x}{\sigma}\right)\left(2-\frac{x}{\sigma}\right)^{\nu-1}, \quad 0 < x < \sigma < \infty. \quad (1.1)$$

For $0 < \nu < 1$, the distribution defined by the density (1.1) is referred to as the J-shaped distribution by Topp and Leone [10] and Nadarajah and

Key words and phrases: Appell's first hypergeometric function, Beta distribution, gamma distribution, Gauss hypergeometric function, Humbert hypergeometric function, Lauricella hypergeometric function, product, quotient, Topp-Leone distribution, transformation.
AMS (MOS) Subject Classifications: 62E15, 62H99.
ISSN 1814-0432, 2025, https://future-in-tech.net

Kotz [5]. For $\nu > 1$, (1.1) attains different shapes depending on values of parameters (see Kotz and van Drop [3]). This family has a close affinity to the family of beta distributions as the distribution of $1-X/\sigma$ has a McDonald beta distribution (Kumaraswamy distribution with parameters 2 and ν) and $(1-X/\sigma)^2$ follows a standard beta distribution with parameters 1 and ν. For $\sigma = 1$, the density in (1.1) reduces to the standard Topp-Leone density and in this case we will write $X \sim \mathrm{TL}(\nu)$.

Several studies focusing on various aspects of the univariate T-L distribution have emerged in recent years, indicating the renewed interest in the Top-Leone family of probability distributions. For further insights into the different facets of T-L distribution and its variations the reader is refereed to Grara and Zghoul [1], Nagar, Zarrazola, and Echeverri-Valencia [6], and Saini, Tomer and Garg [8] and references therein. However, there is relatively little work related to distributions of the product and the quotient of Top-Leone variables.

Distributions of the sum, product, and quotient of random variables (whether correlated or independent from the same or different families) have been the subject of extensive research over the years due to their many useful applications in scientific literature.

In this article, we derive distributions of products and quotients of two independent random variables when at least one of them is Topp-Leone. It is reasonable to think that distributions of product and quotients derived in this article will also find interesting applications and uses in different areas.

This paper is organized as follows. Section 2 gives known definitions and results used in this article. Section 3 is devoted to the derivations for the pdfs of products, while Section 4 deals with the derivations of the pdfs of quotients for beta and Topp-Leone variables.

2 Some definitions

In this section, we will provide definitions and results that will be applied in subsequent sections.

The integral representation of the Gauss hypergeometric function is given as

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}\,\mathrm{d}t, \tag{2.1}$$

where $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$, $|\arg(1-z)| < \pi$. Note that, by expanding

$(1 - zt)^{-b}$, $|zt| < 1$, in (2.1) and integrating t the series expansion for F can be obtained.
The Lauricella hypergeometric function F_D has integral representation

$$F_D^{(n)}(a, b_1, \ldots, b_n; c; z_1, \ldots, z_n) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{u^{a-1}\,(1-u)^{c-a-1}\,\mathrm{d}u}{\prod_{i=1}^{n}(1-uz_i)^{b_i}}, \quad (2.2)$$

where $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$ and $|\arg(1 - z_i)| < \pi$, $i = 1, \ldots, n$. The series form of F_D is

$$\begin{aligned} & F_D^{(n)}(a, b_1, \ldots, b_n; c; z_1, \ldots, z_n) \\ & = \sum_{j_1,\ldots,j_n=0}^{\infty} \frac{(a)_{j_1+\cdots+j_n}(b_1)_{j_1}\cdots(b_n)_{j_n}}{(c)_{j_1+\cdots+j_n}} \frac{z_1^{j_1}\cdots z_n^{j_n}}{j_1!\cdots j_n!}, \quad \max\{|z_1|, \ldots, |z_n|\} < 1, \end{aligned}$$

where the Pochhammer symbol $(a)_n$ is defined by $(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \ldots$ and $(a)_0 = 1$. If $n = 2$, then the Lauricella hypergeometric function $F_D^{(n)}$ slides to Appell's first hypergeometric function F_1, that is, $F_D^{(2)} \equiv F_1$. For $n = 1$, it reduces to the Gauss hypergeometric function.

For further results and properties of these functions the reader is referred to Luke [4], Prudnikov, Brychkov and Marichev [7, 7, Sec. 7.2.4], and Srivastava and Karlsson [9].

Finally, we define the beta type 1 distribution whose definition can be fount in any statistics text.

Definition 2.1. *The random variable X is said to have a beta type 1 distribution with parameters (a, b), $a > 0$, $b > 0$, denoted as $X \sim \mathrm{B1}(a, b)$, if its pdf is given by*

$$\{B(a, b)\}^{-1} x^{a-1}(1 - x)^{b-1}, \quad 0 < x < 1,$$

where

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

The matrix variate generalizations of the gamma, the beta type 1 and the beta type 2 distributions have been defined and studied extensively. For example, see Gupta and Nagar [2].

3 Distribution of the product

In this section we obtain distributional results for the product of two independent random variables involving Topp-Leone distribution.

Theorem 3.1. *Let X_1 and X_2 be independent, $X_i \sim \mathrm{TL}(\nu_i)$, $i = 1, 2$. Then, the pdf of $P = X_1X_2$ is given by*

$$\frac{2}{3}\nu_1\nu_2 p^{\nu_1-1}(2-p)^{\nu_1-1}(1-p)^3$$
$$\times F_D^{(3)}\left(2, 2\nu_1-\nu_2+1, 1-\nu_1, 1-\nu_2; 4; 1-p, \frac{2(1-p)}{2-p}, -(1-p)\right), \; 0 < p < 1.$$

Proof. Using the independence, the joint pdf of X_1 and X_2 is given by

$$4\nu_1\nu_2 x_1^{\nu_1-1}(1-x_1)(2-x_1)^{\nu_1-1}x_2^{\nu_2-1}(1-x_2)(2-x_2)^{\nu_2-1}, \tag{3.1}$$

where $0 < x_1, x_2 < 1$. Transforming $P = X_1X_2$, $V = X_2$ with the Jacobian $J(x_1, x_2 \to p, v) = 1/v$, we obtain the joint pdf of P and V as

$$4\nu_1\nu_2 p^{\nu_1-1}v^{\nu_2-2\nu_1-1}(v-p)(2v-p)^{\nu_1-1}(1-v)(2-v)^{\nu_2-1}, \tag{3.2}$$

where $0 < p < v < 1$. To find the marginal pdf of P, we integrate (3.2) with respect to v to get

$$4\nu_1\nu_2 p^{\nu_1-1}\int_p^1 v^{\nu_2-2\nu_1-1}(v-p)(2v-p)^{\nu_1-1}(1-v)(2-v)^{\nu_2-1}\,\mathrm{d}v$$
$$= 4\nu_1\nu_2 p^{\nu_1-1}(1-p)^3(2-p)^{\nu_1-1}\int_0^1 w(1-w)[1-(1-p)w]^{\nu_2-2\nu_1-1}$$
$$\times\left[1-\frac{2(1-p)}{2-p}w\right]^{\nu_1-1}[1+(1-p)w]^{\nu_2-1}\,\mathrm{d}w,$$

where we have used the change of variable $w = (1-v)/(1-p)$. Finally, applying the definition of F_D given in (2.2), we obtain the desired result. □

Theorem 3.2. *Let X_1 and X_2 be independent random variables, $X_1 \sim \mathrm{TL}(\nu)$ and $X_2 \sim \mathrm{B1}(\alpha, \beta)$. Then, the pdf of $P = X_1X_2$ is*

$$\frac{2\nu\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta+2)}p^{\nu-1}(1-p)^{\beta+1}(2-p)^{\nu-1}$$
$$\times F_1\left(\beta, 2\nu-\alpha+1, -\nu+1; \beta+2; 1-p, \frac{2(1-p)}{2-p}\right),$$

where $0 < p < 1$*. Moreover, if* $\alpha = 2\nu + 1$*, then* P *has the pdf*

$$\frac{2\nu\Gamma(2\nu+\beta+1)}{\Gamma(2\nu+1)\Gamma(\beta+2)} p^{\nu-1}(1-p)^{\beta+1}(2-p)^{\nu-1} F\left(\beta, -\nu+1; \beta+2; \frac{2(1-p)}{2-p}\right).$$

Proof. The joint pdf of X_1 and X_2 is given by

$$\frac{2\nu}{B(\alpha,\beta)} x_1^{\nu-1}(1-x_1)(2-x_1)^{\nu-1} x_2^{\alpha-1}(1-x_2)^{\beta-1}, \quad 0 < x_1, x_2 < 1. \tag{3.3}$$

Transform $(P, V) = (X_1X_2, X_2)$ with the Jacobian $J(x_1, x_2 \to p, v) = 1/v$. Under this transformation, the joint pdf of P and V is

$$\frac{2\nu}{B(\alpha,\beta)} p^{\nu-1} v^{\alpha-2\nu-1}(v-p)(2v-p)^{\nu-1}(1-v)^{\beta-1}, \quad 0 < p < v < 1. \tag{3.4}$$

By integrating (3.4) with respect to v, the marginal pdf of P is obtained as

$$\begin{aligned}
&\frac{2\nu}{B(\alpha,\beta)} p^{\nu-1} \int_p^1 v^{\alpha-2\nu-1}(v-p)(2v-p)^{\nu-1}(1-v)^{\beta-1}\,\mathrm{d}v \\
&= \frac{2\nu}{B(\alpha,\beta)} p^{\nu-1}(1-p)^{\beta+1}(2-p)^{\nu-1} \\
&\quad \times \int_0^1 w^{\beta-1}(1-w)[1-(1-p)w]^{\alpha-2\nu-1}\left[1-\frac{2(1-p)}{2-p}w\right]^{\nu-1}\mathrm{d}w, \qquad (3.5)
\end{aligned}$$

where we have used the substitution $w = (1-v)/(1-p)$. Finally, applying the definition of F_D in (3.5), we obtain the desired result. □

4 Distribution of the quotient

In this section we obtain distributional results for the quotient of two independent random variables involving Topp-Leone distribution.

In the following theorem, we consider the case where both the random variables are distributed as Topp-Leone.

Theorem 4.1. *Let the random variables* X_1 *and* X_2 *be independent,* $X_i \sim \mathrm{TL}(\nu_i)$*,* $i = 1, 2$*. Then, the pdf of* $Z = X_1/X_2$ *is given by*

$$\begin{aligned}
&\frac{2^{\nu_1+\nu_2}\nu_1\nu_2\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1+\nu_2+2)} z^{\nu_1-1}\Bigg[F_1\left(\nu_1+\nu_2, -\nu_1+1, -\nu_2+1; \nu_1+\nu_2+2; \frac{z}{2}, \frac{1}{2}\right) \\
&\quad - \frac{\nu_1+\nu_2}{\nu_1+\nu_2+2}\, z\, F_1\left(\nu_1+\nu_2+1, -\nu_1+1, -\nu_2+1; \nu_1+\nu_2+3; \frac{z}{2}, \frac{1}{2}\right)\Bigg]
\end{aligned}$$

for $0 < z \le 1$, *and*

$$\frac{2^{\nu_1+\nu_2}\nu_1\nu_2\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1+\nu_2+2)} z^{-\nu_2-1}\left[F_1\left(\nu_1+\nu_2,-\nu_1+1,-\nu_2+1;\nu_1+\nu_2+2;\frac{1}{2},\frac{1}{2z}\right)\right.$$
$$\left. -\frac{\nu_1+\nu_2}{\nu_1+\nu_2+2}\ \frac{1}{z}F_1\left(\nu_1+\nu_2+1,-\nu_1+1,-\nu_2+1;\nu_1+\nu_2+3;\frac{1}{2},\frac{1}{2z}\right)\right]$$

for $z > 1$.

Proof. The joint pdf of X_1 and X_2 is given by (3.1). Consider the transformation $Z = X_1/X_2$, $V = X_2$ whose Jacobian is $J(x_1, x_2 \to z, v) = v$. Thus, using (3.1), we obtain the joint pdf of Z and V as

$$4\nu_1\nu_2 z^{\nu_1-1} v^{\nu_1+\nu_2-1}(1-zv)(2-zv)^{\nu_1-1}(1-v)(2-v)^{\nu_2-1}, \tag{4.1}$$

where $0 < v < 1$ for $0 < z \le 1$, and $0 < v < 1/z$ for $z > 1$. For $0 < z \le 1$, the marginal pdf of Z is obtained by integrating (4.1) over $0 < v < 1$. Thus, the pdf of Z, for $0 < z \le 1$, is obtained as

$$2^{\nu_1+\nu_2}\nu_1\nu_2 z^{\nu_1-1}\int_0^1 v^{\nu_1+\nu_2-1}(1-v)(1-zv)\left(1-\frac{zv}{2}\right)^{\nu_1-1}\left(1-\frac{v}{2}\right)^{\nu_2-1}\,\mathrm{d}v.$$

Evaluating the integral by using the integral representation of the Lauricella hypergeometric function, we get the density. For $z > 1$, the density of z is given by

$$2^{\nu_1+\nu_2}\nu_1\nu_2 z^{\nu_1-1}\int_0^{1/z} v^{\nu_1+\nu_2-1}(1-v)(1-zv)\left(1-\frac{1}{2}zv\right)^{\nu_1-1}\left(1-\frac{1}{2}v\right)^{\nu_2-1}\mathrm{d}v$$
$$= 2^{\nu_1+\nu_2}\nu_1\nu_2 z^{-\nu_2-1}\int_0^1 w^{\nu_1+\nu_2-1}(1-w)\left(1-\frac{w}{z}\right)\left(1-\frac{w}{2}\right)^{\nu_1-1}\left(1-\frac{w}{2z}\right)^{\nu_2-1}\mathrm{d}w$$

where we have used the substitution $w = vz$. Finally, using the integral representation of the Lauricella hypergeometric function, we obtain the pdf of Z for $z > 1$. □

Next, we consider the case where independent random variables follow Topp-Leone and beta distributions.

Theorem 4.2. *Let the random variables* X_1 *and* X_2 *be independent,* $X_1 \sim \mathrm{TL}(\nu)$ *and* $X_2 \sim \mathrm{B1}(\alpha,\beta)$*. Then, the pdf of* $Z = X_1/X_2$ *is given by*

$$\frac{2^{\nu}\nu\Gamma(\nu+\alpha)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma((\alpha+\beta+\nu)} z^{\nu-1}\left[F\left(\nu+\alpha,1-\nu;\nu+\alpha+\beta;\frac{z}{2}\right)\right.$$
$$\left. -\frac{\nu+\alpha}{\nu+\alpha+\beta}\,zF\left(\nu+\alpha+1,1-\nu;\nu+\alpha+\beta+1;\frac{z}{2}\right)\right],\quad 0 < z \le 1$$

and

$$\frac{2^{\nu}\nu\Gamma(\nu+\alpha)}{B(\alpha,\beta)\Gamma(\nu+\alpha+2)}z^{-\alpha-1}\,F_1\left(\nu+\alpha,1-\nu,1-\beta;\nu+\alpha+2;\frac{1}{2},\frac{1}{z}\right),\quad z>1.$$

Proof. The joint density of X_1 and X_2 is given in (3.3). Consider the transformation $Z = X_1/X_2$, $U = X_2$ whose Jacobian is $J(x_1, x_2 \to z, u) = u$. Thus, using (3.3), we obtain the joint pdf of Z and U as

$$\frac{2\nu}{B(\alpha,\beta)}z^{\nu-1}u^{\nu+\alpha-1}(1-zu)(2-zu)^{\nu-1}(1-u)^{\beta-1} \tag{4.2}$$

where $0 < u < 1$ for $0 < z \leq 1$, and $0 < u < 1/z$ for $z > 1$. For $0 < z \leq 1$, the marginal pdf of Z is obtained by integrating (4.2) over $0 < u < 1$. Thus, the pdf of Z, for $0 < z \leq 1$, is obtained as

$$\frac{2^{\nu}\nu}{B(\alpha,\beta)}z^{\nu-1}\int_0^1 u^{\nu+\alpha-1}(1-u)^{\beta-1}(1-zu)\left(1-\frac{zu}{2}\right)^{\nu-1}\,\mathrm{d}u.$$

Now, by using the definition of the Gauss hypergeometric function given in (2.1) the required density is obtained. For $z > 1$, the density of z is given by

$$\begin{aligned}&\frac{2\nu}{B(\alpha,\beta)}z^{\nu-1}\int_0^{1/z} u^{\nu+\alpha-1}(1-u)^{\beta-1}(1-zu)\,(2-zu)^{\nu-1}\,\mathrm{d}u\\ &=\frac{2^{\nu}\nu}{B(\alpha,\beta)}z^{-\alpha-1}\int_0^1 w^{\nu+\alpha-1}(1-w)\left(1-\frac{w}{z}\right)^{\beta-1}\left(1-\frac{w}{2}\right)^{\nu-1}\,\mathrm{d}w,\end{aligned}$$

where the last line has been obtained by substituting $w = uz$. Finally, using the integral representation of the Lauricella hypergeometric function, we obtain the pdf of Z for $z > 1$. □

Theorem 4.3. *Let the random variables X_1 and X_2 be independent, $X_i \sim \mathrm{TL}(\nu_i)$, $i = 1, 2$. Then, the pdf of $R = X_1/(X_1 + X_2)$ is given by*

$$\begin{aligned}&\frac{2^{\nu_1+\nu_2}\nu_1\nu_2\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1+\nu_2+2)}r^{\nu_1-1}(1-r)^{-\nu_1-1}\\ &\times\left[F_1\left(\nu_1+\nu_2,1-\nu_1,1-\nu_2;\nu_1+\nu_2+2;\frac{r}{2(1-r)},\frac{1}{2}\right)\right.\\ &\left.-\frac{(\nu_1+\nu_2)r}{(\nu_1+\nu_2+2)(1-r)}F_1\left(\nu_1+\nu_2+1,1-\nu_1,1-\nu_2;\nu_1+\nu_2+3;\frac{r}{2(1-r)},\frac{1}{2}\right)\right]\end{aligned}$$

for $0 < r \leq 1/2$, *and*

$$\frac{2^{\nu_1+\nu_2}\nu_1\nu_2\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1+\nu_2+2)} r^{-\nu_2-1}(1-r)^{\nu_2-1}$$
$$\times \left[F_1\left(\nu_1+\nu_2, 1-\nu_1, 1-\nu_2; \nu_1+\nu_2+2; \frac{1}{2}, \frac{1-r}{2r}\right)\right.$$
$$\left. - \frac{(\nu_1+\nu_2)(1-r)}{(\nu_1+\nu_2+2)r} F_1\left(\nu_1+\nu_2+1, 1-\nu_1, 1-\nu_2; \nu_1+\nu_2+3; \frac{1}{2}, \frac{1-r}{2r}\right)\right]$$

for $1/2 < r < 1$.

Proof. Note that $R = X_1/(X_1+X_2) = Z/(1+Z)$, where $Z = X_1/X_2$ and the density of Z is given in Theorem 4.3. Therefore transforming $R = Z/(1+Z)$ with the Jacobian $J(z \to r) = (1-r)^{-2}$ in Theorem 4.3, we get the density of R. □

Theorem 4.4. *Let the random variables* X_1 *and* X_2 *be independent,* $X_1 \sim \mathrm{TL}(\nu)$ *and* $X_2 \sim \mathrm{B1}(\alpha, \beta)$*. Then, the pdf of* $T = X_1/(X_1+X_2)$ *is given by*

$$\frac{2^{\nu}\nu B(\nu+\alpha,\beta) t^{\nu-1}(1-t)^{-\nu-1}}{B(\alpha,\beta)}\left[F\left(\nu+\alpha, 1-\nu; \nu+\alpha+\beta; \frac{t}{2(1-t)}\right)\right.$$
$$\left. - \frac{\nu+\alpha}{\nu+\alpha+\beta}\frac{t}{1-t} F\left(\nu+\alpha+1, 1-\nu; \nu+\alpha+\beta+1; \frac{t}{2(1-t)}\right)\right]$$

for $0 < t \leq 1/2$, *and*

$$\frac{2^{\nu}\nu\Gamma(\nu+\alpha) t^{-\alpha-1}(1-t)^{\alpha-1}}{B(\alpha,\beta)\Gamma(\nu+\alpha+2)} F_1\left(\nu+\alpha, 1-\nu, 1-\beta; \nu+\alpha+2; \frac{1}{2}, \frac{1-t}{t}\right)$$

for $1/2 < t < 1$.

Proof. We can write $T = Z/(1+Z)$, where $Z = X_1/X_2$ and the density of Z is derived in Theorem 4.2. Therefore, the density of T is obtained by transforming $T = Z/(1+Z)$ with the Jacobian $J(z \to t) = (1-t)^{-2}$ in Theorem 4.2. □

Acknowledgment. This research work was supported by the Sistema Universitario de Investigación, Universidad de Antioquia [project no. 2021-47390].

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