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Neighborhood Systems of Convex Sets in Generalized Fans: A Polynomial Representation

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Abstract

In this work, we characterize the neighborhood systems of convex subsets of the vertex-set of generalized fans. Moreover, we establish its convex neighborhood polynomial.

1 Introduction

Convexity in graphs is a well-studied concept in graph theory. This concept has many applications in optimization problems. On the other hand, the study of representation of graphs using polynomials has captured the interests of researchers because these representations are found to have applications in other sciences [4]. The idea of convex subgraph polynomials

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was pioneered by Artes and Laja [6] in 2014. In 2022, the idea of bivariate polynomial has been introduced by considering the common neighborhood system of cliques in a graph [3] and later extended to clique connected common neighborhood polynomials [2]. Recently, Lahaman et al. [5] established some characterizations of neighborhood systems of convex sets of generalized wheels.

For $u, v \in V(G)$, we define the *geodetic closure* of $\{u, v\}$ as $I_G[u, v] = \{u, v\} \cup \{y : y \text{ lies in a } u\text{-}v \text{ path in } G\}$ [6]. The geodetic closure of $S \subseteq V(G)$ is defined as $I_G[S] = \bigcup_{u,v \in V(G)} I_G[u, v]$ [6]. We say that S is *convex* if $I_G[S] = S$. For a graph G of order n, the *convex neighborhood polynomial* of G, as defined in [1], is the bivariate polynomial given by

$$\Gamma_{cn}(G; x, y) = \sum_{j=0}^{n-i} \sum_{i=1}^{n} c_{ij}(G) x^i y^j,$$

where $c_{ij}(G)$ counts the number of convex subsets S of $V(G)$ satisfying $|S| = i$ and $|N_G(S)| = j$.

2 Main Results

Consider the generalized fan $F_{m,n} = \overline{K_m} \oplus P_n$. From here onward, we set $P_n = [v_1, v_2, \ldots, v_n]$ and $V(\overline{K_m}) = \{u_1, u_2, \ldots, u_m\}$.

Theorem 2.1. [6] *Let G_1 and G_2 be noncomplete graphs. A subset $S = S_1 \cup S_2$ of $V(G_1 \oplus G_2)$, where $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$, is a convex set in $G_1 \oplus G_2$ if and only if S_1 and S_2 induce complete subgraphs in G_1 and G_2, respectively, where it may occur that $S_1 = \varnothing$ or $S_2 = \varnothing$.*

As a corollary, we have:

Corollary 2.2. *Let $m, n \in \mathbb{N}$ satisfy $m, n \geq 3$ and $S \subseteq V(F_{m,n})$. Then S is convex in $F_{m,n}$ iff S satisfies one of the following properties:*

(i) $|S| = 1$.

(ii) $\langle S \rangle = K_2$ *in* P_n

(iii) $S = S_1 \cup S_2$, *where* $S_1 \subseteq V(\overline{K_m})$ *with* $|S_1| = 1$ *and* $\langle S_2 \rangle = K_1$ *in* P_n.

(iv) $S = S_1 \cup S_2$, *where* $S_1 \subseteq V(\overline{K_m})$ *with* $|S_1| = 1$ *and* $\langle S_2 \rangle = K_2$ *in* P_n.

Finally, the polynomial for $F_{m,n}$ is shown below.

Theorem 2.3. *Let $m, n \in \mathbb{N}$ satisfying $m, n \geq 3$. Then*

$$\begin{aligned}\Gamma_{cn}(F_{m,n}; x, y) &= mxy^n + 2xy^{m+1} + (n-2)xy^{m+2} + mnx^2y^{m+n-2} \\ &\quad +2x^2y^{m+1} + (n-3)x^2y^{m+2} + m(n-1)x^3y^{m+n-3}.\end{aligned}$$

Proof. Write the fan as the join of an empty graph of order m and a path of order n as $F_{m,n} = \overline{K_m} \oplus P_n$. Let $S \subseteq V(F_{m,n})$ such that $I_{F_{m,n}}[S] = S$. Then S satisfies the conditions in Lemma 2.2.

Case 1. $|S| = 1$.

Subcase 1.1. $S \subset V(\overline{K_m})$

Let $S = \{u\}$. Then $\deg_{\overline{K_m}}(u) = 0$. Now, for every vertex $v \in V(P_n)$, $uv \in E(F_{m,n})$. Hence, $\deg_{F_{m,n}}(u) = \deg_{\overline{K_m}}(u) + |V(P_n)| = 0 + n = n$. This means that the neighborhood system of S is $V(P_n)$. Hence, $|N_{F_{m,n}}(u)| = n$ and $|N_{F_{m,n}}(S)| = n$

Subcase 1.2. $S \subseteq V(P_n)$.

Note that $\deg_{P_n}(v_1) = \deg_{P_n}(v_n) = 1$ and for every $i \in \{2, 3, \ldots, n-1\}$, $\deg_{P_n}(v_i) = 2$. By the adjacency property of the join $\overline{K_m} \oplus P_n$, $uv_i \in E(F_{m,n})$ for every $u \in V(\overline{K_m})$. Thus, $\deg_{F_{m,n}}(v_1) = \deg_{F_{m,n}}(v_n) = m+1$ and for every $1 \leq i \leq n-1$, $\deg_{F_{m,n}}(v_i) = m+2$.

This case contributes $mxy^n + 2xy^{m+1} + (n-2)xy^{m+2}$.

Case 2. S induces a K_2 in $F_{m,n}$.

Subcase 2.1. $S \subseteq V(P_n)$

If $S = \{v_1, v_2\}$, then $N_{F_{m,n}}(S) = V(\overline{K_m}) \cup \{v_3\}$. Hence, for this convex set, the neighborhood system has cardinality $m+1$. Similarly, for $S = \{v_{n-1}, v_n\}$, we have $N_{F_{m,n}}(S) = V(\overline{K_m}) \cup \{v_{n-2}\}$. Hence, for this convex set, the neighborhood system has cardinality $m+1$. This subcase contributes $2x^2y^{m+1}$.

Now, consider the inner edges of P_n. Let $S = \{v_i, v_{i+1}\}$, where $2 \leq i \leq n-2$. Then $N_{F_{m,n}}(S) = V(\overline{K_m}) \cup \{v_{i-1}, v_{i+2}\}$. Thus, in this case, we have $|N_{F_{m,n}}(S)| = m+2$. This gives $(n-3)x^2y^{m+2}$.

Subcase 2.2. S joins a vertex in $\overline{K_m}$ and a vertex in P_n.

For $1 \leq i \leq n$, let $S = \{u, v_i\}$ with $u \in V(\overline{K_m})$. Note that the neighborhood of u in $F_{m,n}$ is $V(P_n)$ and $V(\overline{K_m}) \subset N_{F_{m,n}}(v_i)$. Hence, the neighborhood system of the set S is the set $V(F_{m,n}) \setminus \{u, v_i\}$. Consequently, $|N_{F_{m,n}}(S)| = m+n-2$. This gives mnx^2y^{m+n-2}.

Case 3. S induces a triangle in $F_{m,n}$.

Let $S = \{u, v_i, v_{i+1}\}$ with $u \in V(\overline{K_m})$ and i ranges from 1 to $n-1$. Then the neighborhood of u in $F_{m,n}$ is $V(P_n)$ and for $1 \leq i \leq n-1$, $V(\overline{K_m}) \subset N_{F_{m,n}}(v_i)$. Hence, $|N_{F_{m,n}}(S)| = m + n - 3$. Clearly, $F_{m,n}$ has $m(n-1)$ triangles. This gives the term $m(n-1)x^3y^{m+n-3}$.

Combining all cases completes the proof. □

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