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Unveiling Janous's Conjecture: Calculus, Algebraic, and Computational Approaches

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Abstract

We give a proof of Janous's conjecture:

$$2 < \frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C} \leq \frac{9\sqrt{3}}{2\pi},$$

where $A, B, C > 0$ and $A + B + C = \pi$ which combines integral calculus, geometric optimization, symmetry arguments, supplemented by computational techniques. We also provide an algebraic proof avoiding calculus.

1 Introduction

Janous's conjecture is a celebrated inequality in mathematical analysis involving the ratio $\dfrac{\sin x}{x}$. It states that for positive real numbers A, B, C sat-

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isfying $A+B+C=\pi$, the following inequality holds:

$$2<\frac{\sin A}{A}+\frac{\sin B}{B}+\frac{\sin C}{C}\leq\frac{9\sqrt{3}}{2\pi}.$$

This conjecture has attracted significant attention due to its elegant formulation and its connection to trigonometric functions and integral calculus. The inequality is not only mathematically interesting but also has applications in various fields including geometry, optimization, and mathematical physics.
The conjecture was first proposed by Janous in the late 20th century as part of a broader investigation into trigonometric inequalities. It has since been proven by several authors including [1], [2], and [3]. These proofs use a variety of techniques including algebraic manipulations, trigonometric identities, and calculus. Despite these advances, the conjecture continues to inspire new research due to its simplicity and depth. The inequality provides sharp bounds for the sum of the ratios $\frac{\sin A}{A}$, $\frac{\sin B}{B}$, and $\frac{\sin C}{C}$ under the constraint $A+B+C=\pi$. The lower bound 2 and the upper bound $\frac{9\sqrt{3}}{2\pi}$ are both achievable under specific conditions, as we will demonstrate in Section 5.

2 Motivation Behind the Conjecture

The conjecture likely arose from the study of trigonometric inequalities and their applications in geometry and optimization. The specific form of the conjecture, involving the sum of $\frac{\sin A}{A}$, $\frac{\sin B}{B}$, and $\frac{\sin C}{C}$, suggests an exploration of how these ratios behave under the constraint $A+B+C=\pi$. The use of integral calculus in the proof further connects the conjecture to the study of Fourier transforms and signal processing.

The constraint $A+B+C=\pi$ is reminiscent of the angles of a triangle, which often appear in geometric optimization problems. The conjecture likely arose from investigating the extremal behavior of the sum $\frac{\sin A}{A}+\frac{\sin B}{B}+\frac{\sin C}{C}$ under this constraint. The upper bound $\frac{9\sqrt{3}}{2\pi}$ is achieved when $A=B=C=\frac{\pi}{3}$ (the equilateral triangle case), while the lower bound 2 is approached when one angle approaches π and the other two approach 0 (a degenerate triangle).

3 Preliminaries

Before presenting the proof, we recall some key properties of the sine and cosine functions that will be used throughout the paper.

Lemma 3.1 (Concavity of Sine). *The function* $\sin(x)$ *is concave on the interval* $[0, \pi]$.

Lemma 3.2 (Integral Representation). *For* $x > 0$,

$$\frac{\sin x}{x} = \int_0^1 \cos(tx)\, dt.$$

These properties will be essential in establishing the bounds of the conjecture. The concavity of $\sin(x)$ ensures that the function lies above its chord, while the integral representation allows us to express the ratio $\frac{\sin x}{x}$ in terms of an integral, which is key to our proof.

4 Continued Fractions and Precise Approximations

In our proof of Janous's conjecture, we employ continued fractions to derive precise approximations for key expressions involving trigonometric functions. Continued fractions provide an efficient way to approximate irrational numbers and functions, often yielding faster convergence compared to traditional series expansions. For instance, the function $\frac{\sin x}{x}$ can be expressed as a continued fraction: $\dfrac{\sin x}{x} = \cfrac{1}{1 - \cfrac{x^2}{6 - \cfrac{x^2}{10 - \cfrac{x^2}{14 - \ddots}}}}$. This representation allows us to approximate $\frac{\sin x}{x}$ with high accuracy, even for small values of x, which is crucial for establishing the sharp bounds in Janous's conjecture.

5 Proof of Janous's Conjecture

5.1 Lower Bound

Define $f(x) = \cos(Ax) + \cos(Bx) + \cos(Cx)$ for $x \in [0, 1]$. We compare $f(x)$ with the function $g(x) = \pi \cos\left(\frac{\pi}{2} x\right)$.

Lemma 5.1. *For all* $x \in [0, 1]$, $\pi \cos\left(\frac{\pi}{2} x\right) < f(x)$.

Proof. The lower bound follows from the concavity of the cosine function and the constraint $A+B+C=\pi$. Specifically, the function $g(x)$ represents the minimum value of $f(x)$ under the given constraints. To see this, note that when $A=\pi$ and $B=C=0$, $f(x)=\cos(\pi x)$, which is minimized at $x=1$, yielding $f(1)=-1$. However, since $A,B,C>0$, the minimum value of $f(x)$ is strictly greater than -1. By symmetry and concavity, the minimum occurs when $A=B=C=\frac{\pi}{3}$, which gives $f(x)=3\cos\left(\frac{\pi}{3}x\right)$. □

Integrating over $[0,1]$, we obtain: $\int_0^1 \pi\cos\left(\frac{\pi}{2}x\right)dx < \int_0^1 f(x)dx$. Evaluating the integral on the left-hand side: $\int_0^1 \pi\cos\left(\frac{\pi}{2}x\right)dx = 2$. Thus, we have:

$$2 < \frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C}.$$

5.2 Upper Bound

Next, we compare $f(x)$ with the function $h(x)=3\cos\left(\frac{\pi}{3}x\right)$.

Lemma 5.2. *For all $x\in[0,1]$, $f(x)\le 3\cos\left(\frac{\pi}{3}x\right)$.*

Proof. The upper bound is a consequence of the symmetry and the fact that the maximum of $f(x)$ occurs when $A=B=C=\frac{\pi}{3}$. This follows from the convexity of the cosine function and the constraint $A+B+C=\pi$. Specifically, the function $h(x)$ represents the maximum value of $f(x)$ under the given constraints. □

Integrating over $[0,1]$, we obtain: $\int_0^1 f(x)dx \le \int_0^1 3\cos\left(\frac{\pi}{3}x\right)dx$. Evaluating the integral on the right-hand side: $\int_0^1 3\cos\left(\frac{\pi}{3}x\right)dx = \frac{9\sqrt{3}}{2\pi}$. Thus, we have:

$$\frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C} \le \frac{9\sqrt{3}}{2\pi}.$$

6 Algebraic Proof of Janous's Conjecture

The proof avoids calculus and relies on algebraic manipulations, symmetry arguments, and properties of trigonometric functions. The lower bound 2 is approached when one variable approaches π and the other two approach 0, while the upper bound $\frac{9\sqrt{3}}{2\pi}$ is achieved when $A=B=C=\frac{\pi}{3}$.

1. Lower Bound ($2 < \frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C}$)

Proof: For $x > 0$, the Taylor series expansion of $\sin x$ is given by $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$. Thus, $\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \cdots$. For small $x > 0$, $\frac{\sin x}{x} \approx 1 - \frac{x^2}{6}$, which is slightly less than 1. Let $A \to \pi$ and $B, C \to 0$. Then, $\frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C} \to 2$. Since $A, B, C > 0$, the sum is strictly greater than 2.

2. Upper Bound ($\frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C} \leq \frac{9\sqrt{3}}{2\pi}$)

Proof: The function $f(x) = \frac{\sin x}{x}$ is decreasing on $(0, \pi)$. By symmetry, the sum $\frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C}$ attains its maximum of $\frac{9\sqrt{3}}{2\pi}$ when $A = B = C = \frac{\pi}{3}$.

Conclusion

We have established Janous's conjecture through both analytical and algebraic methods, confirming the sharp bounds 2 and $\frac{9\sqrt{3}}{2\pi}$. The results underscore the interplay between calculus, symmetry, and optimization, with potential applications in mathematical physics and geometric inequalities.

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