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The Spectral of Convex Bounded Linear Operators on Convex Normed Spaces

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Abstract

We introduce the concept of a convex bounded operator and we define the convex norm for convex bounded linear operator. The aim of this research is to introduce the properties of the spectrum and resolvent sets of convex bounded linear operators on convex normed spaces.

1 Introduction

Alsaffar and Kider [1] introduced the concept of convex normed spaces with its basic properties. Daher and Kider [2] introduced the concept of convex fuzzy normed spaces with its basic properties. Rasha and Kider [3] introduced further properties of the fuzzy complete a-fuzzy normed algebra. Then, Kider [4] introduced the concept of the convex fuzzy metric space and proved its basic properties. Eidi, Hameed and Kider [5] introduced the convex fuzzy distance between two convex fuzzy compact sets.

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2 Convex normed spaces

Definition 2.1. *If the function* $\mathbb{A}_{\mathbb{R}} : \mathbb{R} \to [0, \infty)$ *satisfies*

(i) $\mathbb{A}_{\mathbb{R}}(\lambda) \in [0, \infty),\ \mathbb{A}_{\mathbb{R}}(\lambda) = 0 \iff \lambda = 0.$

(ii) $\mathbb{A}_{\mathbb{R}}(\lambda\delta) = \mathbb{A}_{\mathbb{R}}(\lambda) \cdot \mathbb{A}_{\mathbb{R}}(\delta).$

(iii) $\mathbb{A}_{\mathbb{R}}(\lambda + \delta) \leq \sigma\mathbb{A}_{\mathbb{R}}(\lambda) + \mu\mathbb{A}_{\mathbb{R}}(\delta).$ $\forall\ 0 < \sigma, \mu < 1$, *with* $\sigma + \mu = 1$ *and* $\forall\ \lambda, \delta \in \mathbb{R}$,

then $(\mathbb{R}, \mathbb{A}_{\mathbb{R}})$ *is a convex absolute value space (or c-AVS).*

Definition 2.2. *Let* $\mathbb{V}$ *be an* $\mathbb{R}$*-space over* $\mathbb{R}$, $(\mathbb{R}, \mathbb{A}_{\mathbb{R}})$ *be a c-AVS, and* $\mathcal{N} : \mathbb{V} \to [0, \infty)$ *is a function. If* $\mathcal{N}$ *satisfies:*

(i) $0 \leq \mathcal{N}(y) < \infty$.

(ii) $\mathcal{N}(y) = 0$ *if and only if* $y = 0$.

(iii) $\mathcal{N}(\lambda y) = \mathbb{A}_{\mathbb{R}}(\lambda)\mathcal{N}(y),\ \forall\lambda \in \mathbb{R},\ \lambda \neq 0.$

(iv) $\mathcal{N}(y + g) \leq \gamma\mathcal{N}(y) + \delta\mathcal{N}(g)$, *where* $\gamma, \delta \in (0, 1)$, $\gamma + \delta = 1$, $\forall y, g \in \mathbb{V}$,

then $(\mathbb{V}, \mathcal{N})$ *is a convex normed space (or c-NS).*

Theorem 2.3. *If* $(\mathbb{V}_1, \mathcal{N}_1)$ *and* $(\mathbb{V}_2, \mathcal{N}_2)$ *are two c-NS, then* $(\mathbb{V}, \mathcal{N})$ *is a c-NS, where* $\mathbb{V} = \mathbb{V}_1 \times \mathbb{V}_2$ *and* $\mathcal{N}[(y_1, y_2)] = \gamma\mathcal{N}_1(y_1) + \delta\mathcal{N}_2(y_2)$ *for all* $(y_1, y_2) \in \mathbb{V}$, *for all* $\gamma, \delta \in (0, 1)$ *with* $\gamma + \delta = 1$.

Definition 2.4. *Let* $(\mathbb{V}, \mathcal{N})$ *be a c-NS.*

(i) For any $y \in \mathbb{V}$, *let* $c - \mathbb{B}(y, \alpha) = \{v \in \mathbb{V} : \mathcal{N}(y - v) < \alpha\}$. *Then* $c - \mathbb{B}(y, \alpha)$ *is a convex open ball with center* $y \in \mathbb{V}$ *and radius* $\alpha > 0$.

(ii) $\mathbb{W} \subseteq \mathbb{V}$ *is a convex open set (or simply c-OS) if* $c - \mathbb{B}(w, \alpha) \subseteq \mathbb{W}$ *for any* $w \in \mathbb{W}$ *and for some* $\alpha > 0$.

Definition 2.5. *Let* $(\mathbb{V}, \mathcal{N})$ *be c-NS. If* $(y_k) \in \mathbb{V}$, *then we say that* (y_k) *is convex convergent to* $y \in \mathbb{V}$ *(in notation,* $u_k \to u$ *when* $k \to \infty$*) if* $\forall c - \mathbb{B}(y, \alpha)$ $\exists M$ *such that* $y_k \in c\text{-}\mathbb{B}(y, \alpha)$, $\forall k \geq M$ *where* $\alpha > 0$. *This is equivalent to saying that* $\forall\alpha > 0$ $\exists M \in \mathbb{N}$ *satisfies* $\mathcal{N}(y_k - y) < \alpha$, $\forall k \geq M$. *We also use the notationa* $\lim_{k\to\infty} y_k = y$ *or* $\lim_{n\to\infty} \mathcal{N}(y_k - y) = 0$.

Definition 2.6. *If* $(\mathbb{V}, \mathcal{N})$ *is c-NS, then* *(i)* $\mathbb{Y} \subset \mathbb{V}$ *is convex bounded (or simply CB) if* $\exists\lambda > 0$ *such that* $\mathcal{N}(p) < \mu$, $\forall p \in \mathbb{Y}$. *Otherwise,* $\mathbb{Y}$ *is not CB.*

(ii) *a sequence* (u_k) *in a c-NS* $(\mathbb{V}, \mathcal{N})$ *is CB if* $\exists\lambda > 0$ *such that* $\mathcal{N}(u_k) < \lambda$, $\forall k \in \mathbb{N}$. *Otherwise,* (u_k) *is not CB.*

Definition 2.7. *Let* $(\mathbb{V}, \mathcal{N})$ *be c-NS and let* $(y_k) \in \mathbb{V}$. *We say that* (y_k) *is convex Cauchy in* $\mathbb{V}$ *if* $\forall\sigma > 0$, $\exists M \in \mathbb{N}$ *such that* $\mathcal{N}(y_j - y_p) < \sigma$, $\forall j, p \geq M$.

Definition 2.8. *Let* $(\mathbb{V}, \mathcal{N})$ *be c-NS.* $(\mathbb{V}, \mathcal{N})$ *is called convex complete if* $\forall$ *convex Cauchy sequence* $(y_k) \in \mathbb{V}$, $\exists y \in \mathbb{V}$, *such that* $y_k \to y$.

Theorem 2.9. *If* $(\mathbb{V}_1, \mathcal{N}_1)$ *and* $(\mathbb{V}_2, \mathcal{N}_2)$ *are two c-NS, then* $(\mathbb{V}, \mathcal{N})$ *is convex complete if and only if* $(\mathbb{V}_1, \mathcal{N}_1)$ *and* $(\mathbb{V}_2, \mathcal{N}_2)$ *are convex complete, where* $\mathbb{V} = \mathbb{V}_1 \times \mathbb{V}_2$ *and* $\mathcal{N}[(v_1, v_2)] = \gamma\mathcal{N}_1(v_1) + \delta\mathcal{N}_2(v_2)$ *for all* $(v_1, v_2) \in \mathbb{V}$ *where* $\gamma + \delta = 1$.

Definition 2.10. *If* $(\mathbb{V}, \mathcal{N}_{\mathbb{V}})$ *and* $(\mathbb{Y}, \mathcal{N}_{\mathbb{Y}})$ *are two c-NS, then* $\mathcal{T} : \mathbb{V} \to \mathbb{Y}$ *is convex continuous at* $v \in \mathbb{V}$ *if* $\forall\alpha > 0$, $\exists\beta > 0$, $\mathcal{N}_{\mathbb{V}}(v - y) < \beta$ *implies* $\mathcal{N}_{\mathbb{Y}}[\mathcal{T}(v) - \mathcal{T}(y)] < \alpha$, $\forall y \in \mathbb{Y}$. *If this is true* $\forall v \in \mathbb{V}$, *then* $\mathbb{T}$ *is convex continuous on* $\mathbb{V}$.

Theorem 2.11. *If* $(\mathbb{V}, \mathcal{N}_{\mathbb{V}})$ *and* $(\mathbb{Y}, \mathcal{N}_{\mathbb{Y}})$ *are two c-NS, then the operator* $\mathcal{T} : \mathbb{V} \to \mathbb{Y}$ *is convex continuous at* $y \in \mathbb{V}$ *if and only if* $y_k \to y \in \mathbb{V}$ *implies* $\mathcal{T}(y_k) \to \mathcal{T}(y) \in \mathbb{Y}$.

Theorem 2.12. *If* $(\mathbb{V}, \mathcal{N}_{\mathbb{V}})$ *and* $(\mathbb{Y}, \mathcal{N}_{\mathbb{Y}})$ *are two c-NS, then (i)* $\Rightarrow$ *(ii)* $\Rightarrow$ *(iii)* $\Rightarrow$ *(i)*

(i) *The operator* $\mathcal{T} : \mathbb{V} \to \mathbb{Y}$ *is convex continuous at* $v \in \mathbb{V}$*;*

(ii) $\mathcal{T}^{-1}(\mathbb{Y})$ *is c-OS in* $\mathbb{V}$ *for all c-OS subset* $\mathbb{Y}$ *of* $\mathbb{Y}$*;*

(iii) $\mathcal{T}^{-1}(\mathbb{Y})$ *is convex closed in* $\mathbb{V}$, $\forall$ *convex closed* $\mathbb{Y} \subset \mathbb{Y}$.

Definition 2.13. *Suppose that* $(\mathbb{U}, \mathcal{N}_{\mathbb{U}})$ *and* $(\mathbb{V}, \mathcal{N}_{\mathbb{V}})$ *are c-NS.* $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ *is convex bounded (or CB) if* $\exists\mu > 0$ *and for each* $y \in \mathbb{U}$, *such that*

$$\mathcal{N}_{\mathbb{V}}[\mathbb{T}(y)] < \mu\mathcal{N}_{\mathbb{U}}(y) \quad \ldots \quad (1)$$

Notation 2.14. *Suppose that* $(\mathbb{U}, \mathcal{N}_{\mathbb{U}})$ *and* $(\mathbb{V}, \mathcal{N}_{\mathbb{V}})$ *are two c-NS. Put* $CB(\mathbb{U}, \mathbb{V}) = \{\mathbb{T} : \mathbb{U} \to \mathbb{V} : \mathbb{T}$ *is a linear CB operator*$\}$.

Definition 2.15. *Suppose that* $(\mathbb{U}, \mathcal{N}_{\mathbb{U}})$ *and* $(\mathbb{V}, \mathcal{N}_{\mathbb{V}})$ *are two c-NS. Define:* $\mathcal{N}_{CB(\mathbb{U},\mathbb{V})}(\mathbb{T}) = \sup_{y\in\mathbb{U}} \mathcal{N}_{\mathbb{V}}(\mathbb{T}y)$, *for all* $\mathbb{T} \in CB(\mathbb{U}, \mathbb{V})$.

Remark 2.16. *Equation (1) can be written as*

$$\mathcal{N}_{\mathbb{V}}[\mathbb{T}(y)] < \mathcal{N}_{CB(\mathbb{U},\mathbb{U})}[\mathbb{T}] \cdot \mathcal{N}_{\mathbb{U}}(y) \quad \dots \quad (2)$$

Theorem 2.17. *If* $(\mathbb{U}, \mathcal{N}_{\mathbb{U}})$ *and* $(\mathbb{V}, \mathcal{N}_{\mathbb{V}})$ *are two c-NS, then* $(CB(\mathbb{U}, \mathbb{V}), \mathcal{N}_{CB(\mathbb{U},\mathbb{V})})$ *is a c-NS.*

Proof.

(i) Since $0 \leq \mathcal{N}_{\mathbb{V}}(\mathbb{T}y) < \infty$ with $y \in \mathbb{U}$, so $0 \leq \mathcal{N}_{CB(\mathbb{U},\mathbb{V})}(\mathbb{T}) < \infty$.

(ii) $\mathcal{N}_{CB(\mathbb{U},\mathbb{V})}(\mathbb{T}) = 0 \iff \sup_{y \in \mathbb{U}} \mathcal{N}_{\mathbb{V}}(\mathbb{T}y) = 0 \iff \mathcal{N}_{\mathbb{V}}(\mathbb{T}y) = 0$ for all $y \in \mathbb{U} \iff \mathbb{T}y = 0$ for all $y \in \mathbb{U} \iff \mathbb{T} = 0$.

(iii) For all $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, we have $\mathcal{N}_{CB(\mathbb{U},\mathbb{V})}(\alpha\mathbb{T}) = \sup_{y \in \mathbb{U}} \mathcal{N}_{\mathbb{V}}(\alpha\mathbb{T}y) = \mathbb{A}_{\mathbb{R}}(\alpha) \sup_{y \in \mathbb{U}} \mathcal{N}_{\mathbb{V}}(\mathbb{T}y) = \mathbb{A}_{\mathbb{R}}(\alpha)\mathcal{N}_{CB(\mathbb{U},\mathbb{V})}(\mathbb{T})$.

(iv) $\mathcal{N}_{CB(\mathbb{U},\mathbb{V})}[\mathbb{T}_1 + \mathbb{T}_2] = \sup_{y \in \mathbb{U}} \mathcal{N}_{\mathbb{V}}((\mathbb{T}_1 + \mathbb{T}_2)y) \leq \delta_1 \sup_{y \in \mathbb{U}} \mathcal{N}_{\mathbb{V}}(\mathbb{T}_1 y) + \delta_2 \sup_{y \in \mathbb{U}} \mathcal{N}_{\mathbb{V}}(\mathbb{T}_2 y)$, where $\delta_1, \delta_2 \in (0, 1)$ with $\delta_1 + \delta_2 = 1$. Thus, $n_{CB(\mathbb{U},\mathbb{V})}[\mathbb{T}_1 + \mathbb{T}_2] \leq \delta_1 \mathcal{N}_{CB(\mathbb{U},\mathbb{V})}[\mathbb{T}_1] + \delta_2 \mathcal{N}_{CB(\mathbb{U},\mathbb{V})}[\mathbb{T}_2]$.

Hence, $(CB(\mathbb{U}, \mathbb{V}), \mathcal{N}_{CB(\mathbb{U},\mathbb{V})})$ is a c-NS.

Theorem 2.18. *Suppose that* $(\mathbb{U}, \mathcal{N}_{\mathbb{U}})$ *and* $(\mathbb{V}, \mathcal{N}_{\mathbb{V}})$ *are two c-NS. The linear operator* $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ *is convex continuous if and only if* $\mathbb{T}$ *is CB.*

Proof.

If $\mathbb{T}$ is convex continuous at $y \in D(\mathbb{T})$, then $\forall \alpha > 0, \exists \epsilon > 0$ such that when $\mathcal{N}_{\mathbb{V}}[\mathbb{T}z - \mathbb{T}y] < \alpha$, we have $\mathcal{N}_{\mathbb{U}}(z - y) < \alpha$, $\forall z \in D(\mathbb{T})$. If $\mathbb{W} \subseteq D(\mathbb{T})$ is CB, choose $w \neq 0 \in \mathbb{W}$. Put $w = z - y$. Then, $\mathcal{N}_{\mathbb{V}}(\mathbb{T}w) = \mathcal{N}_{\mathbb{V}}(\mathbb{T}(z - y)) < \sigma$. It follows that $\mathbb{T}(\mathbb{W})$ is CB. Hence, $\mathbb{T}$ is CB.
Conversely, assume that $\mathbb{T}$ is CB. If $\alpha > 0$ then $\forall y \in D(\mathbb{T})$ satisfies:
$\mathcal{N}_{\mathbb{U}}(y) < \alpha$ implies $\mathcal{N}_{\mathbb{V}}(\mathbb{T}u) < \sigma$ where $\sigma > 0$.
Thus, for $w \in D(\mathbb{T})$, we have
$\mathcal{N}_{\mathbb{U}}(y - w) < \alpha \Rightarrow \mathcal{N}_{\mathbb{V}}(\mathbb{T}y - \mathbb{T}w) = \mathcal{N}_{\mathbb{V}}(\mathbb{T}(y - w)) < \sigma$.
Hence, $\mathbb{T}$ is convex continuous at y. Since y was any vector in $D(\mathbb{T})$, $\mathbb{T}$ is convex continuous.

Theorem 2.19. *Suppose that* $(\mathbb{U}, \mathcal{N}_{\mathbb{U}})$ *and* $(\mathbb{V}, \mathcal{N}_{\mathbb{V}})$ *are two c-NS. Then* $CB(\mathbb{U}, \mathbb{V})$ *is convex complete if* $\mathbb{V}$ *is convex complete.*

Proof.

The proof is direct.

Definition 2.20. *The linear functional* $f : (\mathbb{U}, \mathcal{N}) \to (\mathbb{R}, \mathbb{A}_{\mathbb{R}})$ *is CB if* $\exists \mu > 0$ *such that* $\mathbb{A}_{\mathbb{R}}[f(y)] < \mu \cdot \mathcal{N}(y), \forall y \in \mathbb{U}$. *Also,* $\mathcal{N}_{CB(\mathbb{U},\mathbb{R})}(f) = \sup_{u \in \mathbb{U}} \mathbb{A}_{\mathbb{R}}(fy), \forall f \in CB(\mathbb{U}, \mathbb{R})$*; that is,* $\mathbb{A}_{\mathbb{R}}[f(y)] < \mathcal{N}_{CB(\mathbb{U},\mathbb{R})}(f) \cdot \mathcal{N}(y), \forall y \in \mathbb{U}$.

Definition 2.21. *If* $(\mathbb{U}, \mathcal{N})$ *is c-NS, then* $(CB(\mathbb{U}, \mathbb{R}), \mathcal{N}_{CB(\mathbb{U},\mathbb{R})})$ *is c-NS, where* $CB(\mathbb{U}, \mathbb{R}) = \{f : \mathbb{U} \to \mathbb{R} : f \text{ is linear CB}\}$ *and* $\mathcal{N}_{CB(\mathbb{U},\mathbb{R})}(f) = \sup_{u \in \mathbb{U}} \mathbb{A}_{\mathbb{R}}(fu)$. *This is called the convex dual space of* $\mathbb{U}$.

By using Theorem 2.19 and the fact that the c-NS $(\mathbb{R}, \mathbb{A}_{\mathbb{R}})$ is convex complete, we have the following theorem.

Theorem 2.22. *If* $(\mathbb{U}, \mathcal{N})$ *is c-NS, then the convex dual space* $CB(\mathbb{U}, \mathbb{R})$ *is convex complete.*

3 The Spectral of CB Linear operators

Definition 3.1. *Let* $(\mathbb{U}, \mathcal{N})$ *be c-NS and let* $\mathbb{S} : D(\mathbb{S}) \to \mathbb{U}$ *be a linear operator. Associate with* $\mathbb{S}$ *the operator* $\mathbb{S}_\alpha = (\mathbb{S} - \alpha\mathbb{I})$*, where* $\alpha \in \mathbb{C}$. *If* $\mathbb{S}_\alpha^{-1}$ *exists, then it is* $\Re_\alpha(\mathbb{S}) = \mathbb{S}_\alpha^{-1} = (\mathbb{S} - \alpha\mathbb{I})^{-1}$. *The operator* $\Re_\alpha(\mathbb{S})$ *is called the resolvent operator (or resolvent) of* $\mathbb{S}$. *For brevity, we write* $\Re_\alpha$ *instead of* $\Re_\alpha(\mathbb{S})$.

Definition 3.2. *Let* $(\mathbb{U}, \mathcal{N})$ *be c-NS and let* $\mathbb{S}$ *is a linear operator from* $D(\mathbb{S}) \subseteq \mathbb{U}$ *into* $\mathbb{U}$. *With* $\alpha \in \mathbb{C}$, *the regular value* α *of* $\mathbb{S}$ *is such that:*

(i) $\Re_\alpha(\mathbb{S})$ *exists.*

(ii) $\Re_\alpha(\mathbb{S})$ *is convex bounded.*

(iii) $\Re_\alpha(\mathbb{S})$ *is defined on* $\mathbb{W} \subseteq D(\mathbb{S})$ *and* $\overline{\mathbb{W}} = \mathbb{U}$.

The resolvent set $\rho(\mathbb{S}) = \{\alpha \in \mathbb{C} : \alpha \text{ is a regular value of } \mathbb{S}\}$ *and* $\rho(\mathbb{S})^C = \mathbb{C} - \rho(\mathbb{S}) = \sigma(\mathbb{S})$ *is called the spectrum of* $\mathbb{S}$. *Also,* $\alpha \in \sigma(\mathbb{S})$ *is called a spectral value of* $\mathbb{S}$.

Remark 3.3. *Let* $(\mathbb{U}, \mathcal{N})$ *be c-NS and let* $\mathbb{S}$ *be a linear operator from* $D(\mathbb{S}) \subseteq \mathbb{U}$ *into* $\mathbb{U}$. *If* $\Re_\alpha(\mathbb{S})$ *exists, then it is linear. Also,* $\Re_\alpha(\mathbb{S}) : \Re(\mathbb{S}_\alpha) \to D(\mathbb{S}_\alpha)$ *exists* $\iff$ $\mathbb{S}_\alpha(z) = 0 \Rightarrow z = 0$*; that is,* $N(\mathbb{S}_\alpha) = \{0\}$*, where* $\Re(\mathbb{S}_\alpha)$ *is the range of* $\mathbb{S}_\alpha$. *Hence, if* $\mathbb{S}_\alpha(z) = (\mathbb{S} - \alpha\mathbb{I})(z) = 0$ *for some* $z \neq 0$*, then* $\alpha \in \sigma_p(\mathbb{S})$*; that is,* α *is an eigenvalue of* $\mathbb{S}$.

Definition 3.4. *Let* $(\mathbb{U},\mathcal{N}_{\mathbb{U}})$*,* $(\mathbb{W},\mathcal{N}_{\mathbb{W}})$ *be two c-NS and let*$\mathbb{S}:\mathbb{U}\to\mathbb{W}$ *be a linear operator. Then* $\mathbb{S}$ *is convex closed if* $G(\mathbb{S})=\{(u,w):u\in\mathbb{U}, w=\mathbb{S}(u)\}$*, the graph of* $\mathbb{S}$*, is convex closed in the c-NS* $(\mathbb{U}\times\mathbb{W},n)$*, where* $\mathcal{N}[(u,w)]=\gamma\mathcal{N}_{\mathbb{U}}(u)+\delta\mathcal{N}_{\mathbb{W}}(w)$ *for all* $(u,w)\in\mathbb{U}\times\mathbb{W}$*, and where* $\gamma,\delta\in(0,1)$ *with* $\gamma+\delta=1$*.*

Theorem 3.5. *Let* $(\mathbb{U},\mathcal{N}_{\mathbb{U}})$*,* $(\mathbb{W},\mathcal{N}_{\mathbb{W}})$ *be two c-NS and the linear operator* $\mathbb{S}:\mathbb{U}\to\mathbb{W}$*.* $\mathbb{S}$ *is convex closed* $\iff$ *if* $u_k\to u$ *with* $u_k\in D(\mathbb{S})$ *and* $\mathbb{S}(u_k)\to w$*, then* $u\in D(\mathbb{S})$ *with* $\mathbb{S}(u)=w$*.*

Proof.
By the definition of $G(\mathbb{S})$, it is convex closed $\iff$ $z=(u,w)\in\overline{G(\mathbb{S})}$ $\iff$ $\exists z_k=(u_k,\mathbb{S}(u_k))\in G(\mathbb{S})$ such that $u_k\to u$, $\mathbb{S}(u_k)\to w$ and $z=(u,w)\in G(\mathbb{S})$ $\iff$ $u\in D(\mathbb{S})$ with $\mathbb{S}(u)=w$.

Theorem 3.6. *Let* $(\mathbb{U},\mathcal{N}_{\mathbb{U}})$*,* $(\mathbb{W},\mathcal{N}_{\mathbb{W}})$ *be two convex complete c-NS and let* $\mathbb{S}:\mathbb{U}\to\mathbb{W}$ *be a convex closed operator. If* $D(\mathbb{S})$ *is convex closed in* $\mathbb{W}$*, then the operator* $\mathbb{S}$ *is CB and* $\mathbb{S}$ *is CB.*

Proof.
$(\mathbb{U}\times\mathbb{W},\mathcal{N})$ is convex complete by Theorem 2.9. If $G(\mathbb{S})$ is convex closed in $\mathbb{U}\times\mathbb{W}$ and $D(\mathbb{S})$ is convex closed in $\mathbb{U}$, then $G(\mathbb{S})$, $D(\mathbb{S})$ are convex complete. Let $\mathbb{Y}:G(\mathbb{S})\to D(\mathbb{S})$ be defined by: $\mathbb{Y}[z,\mathbb{S}(z)]=z$ Then $\mathbb{Y}$ is linear and $\mathbb{Y}$ is CB since $\mathcal{N}(\mathbb{Y}[z,\mathbb{S}(z)])=\mathcal{N}_{\mathbb{U}}(z)\le\gamma\mathcal{N}_{\mathbb{U}}(z)+\delta\mathcal{N}_{\mathbb{W}}[\mathbb{S}(z)]=\mathcal{N}(\mathbb{Y}[z,\mathbb{S}(z)])$, where $\gamma,\delta\in(0,1)$ with $\gamma+\delta=1$. Since $\mathbb{Y}$ is bijective, we can define $\mathbb{Y}^{-1}:D(\mathbb{S})\to G(\mathbb{S})$ as: $\mathbb{Y}^{-1}(z)=[z,\mathbb{S}(z)]$. So, $\mathbb{Y}^{-1}$ is CB, say $\mathcal{N}[z,\mathbb{S}(z)]\le\sigma\mathcal{N}_{\mathbb{U}}(z)$, with $\sigma>0$, $\forall z\in D(\mathbb{S})$. Thus, $\mathbb{S}$ is CB because $\mathcal{N}_{\mathbb{W}}[\mathbb{S}(z)]\le\gamma\mathcal{N}_{\mathbb{U}}(z)+\delta\mathcal{N}_{\mathbb{W}}[\mathbb{S}(z)]=\mathcal{N}[z,\mathbb{S}(z)]\le\sigma\mathcal{N}_{\mathbb{U}}(z)$, for all $z\in D(\mathbb{S})$.

Proposition 3.7. *Let* $\mathbb{S}:\mathbb{U}\to\mathbb{W}$ *be a CB linear operator, where* $(\mathbb{U},\mathcal{N}_{\mathbb{U}})$ *and* $(\mathbb{W},\mathcal{N}_{\mathbb{W}})$ *are two c-NS. Then*

(i) $\mathbb{S}$ *is convex closed if* $D(\mathbb{S})$ *is a convex closed subset of* $\mathbb{U}$*.*

(ii) $D(\mathbb{S})$ *is a convex closed subset of* $\mathbb{U}$ *if* $\mathbb{S}$ *is convex closed and* $(\mathbb{W},\mathcal{N}_{\mathbb{W}})$ *is convex complete.*

Proof.

(i) If $(z_k)\in D(\mathbb{S})$ and $z_k\to z$ and $\mathbb{S}(z_k)\to w$, since $\mathbb{S}$ is convex continuous, then $z\in\overline{D(\mathbb{S})}=D(\mathbb{S})$ because $D(\mathbb{S})$ is convex closed. Hence, $\mathbb{S}$ is convex closed by Theorem 3.6.

(ii) For $z \in \overline{D(\mathbb{S})}$, $\exists (z_k) \in D(\mathbb{S})$, $z_k \to z$ because $\mathbb{S}$ is CB. $\mathcal{N}_{\mathbb{W}}[\mathbb{S}(z_k) - \mathbb{S}(z_m)] = \mathcal{N}_{\mathbb{W}}[\mathbb{S}(z_k - z_m)] \leq \mathcal{N}_{CB(\mathbb{U},\mathbb{W})}(\mathbb{S}) \cdot \mathcal{N}_{\mathbb{U}}[z_k - z_m]$ But this implies that $(\mathbb{S}(z_k))$ is convex Cauchy. So, $\mathbb{S}(z_k) \to w \in \mathbb{W}$ because $\mathbb{W}$ is convex complete. Since $\mathbb{S}$ is convex closed, $z \in D(\mathbb{S})$ by Theorem 3.5 and $\mathbb{S}(z) = w$. Hence, $\overline{D(\mathbb{S})} \subseteq D(\mathbb{S})$. Therefore, $D(\mathbb{S})$ is convex closed.

Lemma 3.8. *Let* $(\mathbb{U}, \mathcal{N})$ *be c-NS and* $\mathbb{S} : \mathbb{U} \to \mathbb{U}$ *be a linear operator and* $\alpha \in \rho(\mathbb{S})$. *If* $\mathbb{S}$ *is convex closed, then* $\Re_\alpha(\mathbb{S})$ *is defined on* $\mathbb{U}$ *and is CB.*

Proof.
Since $\mathbb{S}$ is convex closed, so is $\mathbb{S}_\alpha$ by Theorem 3.6. Hence, $\Re_\alpha$ is convex closed by (*ii*) of Definition 3.2. Thus, its domain $D(\Re_\alpha)$ is convex closed by(ii) Proposition 3.7. Thus (iii) of Definition 3.2 implies $D(\Re_\alpha) = \overline{D(\Re_\alpha)} = \mathbb{U}$.

Lemma 3.9. *Let* $(\mathbb{U}, \mathcal{N})$ *be c-N,S* $\mathbb{S} : \mathbb{U} \to \mathbb{U}$ *be a linear operator, and* $\alpha \in \rho(\mathbb{S})$. *If* $\mathbb{S}$ *is CB, then* $\Re_\alpha(\mathbb{S})$ *is defined on* $\mathbb{U}$ *and is CB.*

Proof.
Since $D(\mathbb{S}) = \mathbb{U}$ is convex closed, $\mathbb{S}$ is convex closed by Theorem 3.6 and the statement follows by Lemma 3.8.

Theorem 3.10. *Let* $(\mathbb{U}, \mathcal{N})$ *be convex complete c-NS and let* $\mathbb{S} : \mathbb{U} \to \mathbb{U}$ *be a linear operator. If* $\mathcal{N}_{CB(\mathbb{U},\mathbb{U})}(\mathbb{S}) < 1$, *then* $(\mathbb{I} - \mathbb{S})^{-1}$ *exists as a CB linear operator on* $\mathbb{U}$ *and* $(\mathbb{I} - \mathbb{S})^{-1} = \sum_{j=1}^{\infty} \mathbb{S}^j$.

Proof.
We know that $\mathcal{N}_{CB(\mathbb{U},\mathbb{U})}(\mathbb{S}^j) \leq [\mathcal{N}_{CB(\mathbb{U},\mathbb{U})}(\mathbb{S})]^j$ for any $j \in \mathbb{N}$. Also, since $\mathcal{N}_{CB(\mathbb{U},\mathbb{U})}(\mathbb{S}) < 1$, the series $\sum_{j=1}^{\infty} [\mathcal{N}_{CB(\mathbb{U},\mathbb{U})}(\mathbb{S})]^j$ is convex convergent. Hence, the series $\sum_{j=1}^{\infty} \mathbb{S}^j$ is absolutely convex convergent for $\mathcal{N}_{CB(\mathbb{U},\mathbb{U})}(\mathbb{S}) < 1$. Since $\mathbb{U}$ is convex complete, so is $CB(\mathbb{U}, \mathbb{U})$ by Theorem 2.19. Let $\mathbb{T} = \sum_{j=1}^{\infty} \mathbb{S}^j$. Now, we will show that $\mathbb{T} = (\mathbb{I} - \mathbb{S})^{-1}$. Consider $(\mathbb{I} - \mathbb{S})(\mathbb{I} + \mathbb{S} + \mathbb{S}^2 + \cdots + \mathbb{S}^n) = (\mathbb{I} + \mathbb{S} + \mathbb{S}^2 + \cdots + \mathbb{S}^n)(\mathbb{I} - \mathbb{S}) = \mathbb{I} - \mathbb{S}^{n+1}$. As $n \to \infty$, we get $\mathbb{S}^{n+1} \to 0$ because $\mathcal{N}_{CB(\mathbb{U},\mathbb{U})}(\mathbb{S}) < 1$. Thus, we have $(\mathbb{I} - \mathbb{S})\mathbb{T} = \mathbb{T}(\mathbb{I} - \mathbb{S})$. Hence, $\mathbb{T} = (\mathbb{I} - \mathbb{S})^{-1}$.

Theorem 3.11. *Let* $(\mathbb{U}, \mathcal{N})$ *be convex complete c-NS and let* $\mathbb{S} : \mathbb{U} \to \mathbb{U}$ *be a linear operator. If* $\mathbb{S}$ *is CB, then the resolvent set* $\rho(\mathbb{S})$ *is a convex open set in* $\mathbb{C}$. *Hence, the spectrum* $\sigma(\mathbb{S})$ *is a convex closed set in* $\mathbb{C}$.

Proof.
If $\rho(\mathbb{S}) = \emptyset$, then it is convex open. Let $\rho(\mathbb{S}) \neq \emptyset$. For fixed $\alpha_0 \in \rho(\mathbb{S})$ and any $\alpha \in \mathbb{C}$, we have $(\mathbb{S} - \alpha\mathbb{I}) = (\mathbb{S} - \alpha_0\mathbb{I} + \alpha_0\mathbb{I} - \alpha\mathbb{I}) =$

$(\mathbb{S}-\alpha_0\mathbb{I})-(\alpha-\alpha_0)\mathbb{I}=(\mathbb{S}-\alpha_0\mathbb{I})[\mathbb{I}-(\alpha-\alpha_0)](\mathbb{S}-\alpha_0\mathbb{I})^{-1}$. Let $H=[\mathbb{I}-(\alpha-\alpha_0)](\mathbb{S}-\alpha_0\mathbb{I})^{-1}$. Then

$$\mathbb{S}_\alpha=\mathbb{S}-\alpha\mathbb{I}=\mathbb{S}_{\alpha_0}H \quad \dots \quad (3)$$

where $H=\mathbb{I}-(\alpha-\alpha_0)\Re_{\alpha_0}$. Since $\alpha_0\in\rho(\mathbb{S})$ and $\mathbb{S}$ is CB, Lemma 3.9 implies that $\Re_{\alpha_0}=\mathbb{S}_{\alpha_0}^{-1}\in CB(\mathbb{U},\mathbb{U})$. Moreover, Theorem 3.10 implies that H has an inverse $H^{-1}=\sum_{j=0}^{\infty}[(\alpha-\alpha_0)\Re_{\alpha_0}]^j=\sum_{j=0}^{\infty}\left[(\alpha-\alpha_0)^j\Re_{\alpha_0}^j\right]$ in $CB(\mathbb{U},\mathbb{U})$ for all α such that $n_{CB(\mathbb{U},\mathbb{U})}((\alpha-\alpha_0)\Re_{\alpha_0})<1$; that is,

$$\mathbb{A}_{\mathbb{C}}[(\alpha-\alpha_0)]<\frac{1}{\mathcal{N}_{CB(\mathbb{U},\mathbb{U})}((\alpha-\alpha_0)\Re_{\alpha_0})} \quad \dots \quad (4)$$

Since $\mathbb{S}_{\alpha_0}^{-1}=\Re_{\alpha_0}\in CB(\mathbb{U},\mathbb{U})$, using this and (3), for all α fulfilling (4), the inverse of $\mathbb{S}_\alpha$ exists

$$\Re_\alpha=\mathbb{S}_\alpha^{-1}=[\mathbb{S}_{\alpha_0}H]^{-1}=H^{-1}\Re_{\alpha_0}. \quad \dots \quad (5)$$

Hence, (4) introduces a convex ball of α_0 consisting of regular values α of $\mathbb{S}$. Since $\alpha_0\in\rho(\mathbb{S})$ was arbitrary, we conclude that $\rho(\mathbb{S})$ is convex open which implies that $\sigma(\mathbb{S})$ is convex closed since $\sigma(\mathbb{S})=\rho(\mathbb{S})^C$. By using equation (5), we immediately have the next result.

Theorem 3.12. *If $(\mathbb{U},\mathcal{N})$ is convex complete c-NS and $\mathbb{S}:\mathbb{U}\to\mathbb{U}$ is a CB linear operator, then for every $\alpha_0\in\rho(\mathbb{S})$, the resolvent $\Re_\alpha(\mathbb{S})$ has the representation $\Re_\alpha=\sum_{j=0}^{\infty}\left[(\alpha-\alpha_0)^j\Re_{\alpha_0}^{j+1}\right] \quad \dots \quad (6)$.*

Proof.
The infinite series $\sum_{j=0}^{\infty}\left[(\alpha-\alpha_0)^j\Re_{\alpha_0}^{j+1}\right]$ is convex absolutely convergent for all α in the convex open ball given by $\mathbb{A}_{\mathbb{C}}[(\alpha-\alpha_0)]<\frac{1}{\mathcal{N}_{CB(\mathbb{U},\mathbb{U})}((\alpha-\alpha_0)\Re_{\alpha_0})}$ in the complex plane. This open ball is a subset of $\rho(\mathbb{S})$.

Theorem 3.13. *If $(\mathbb{U},\mathcal{N})$ is convex complete c-NS and $\mathbb{S}:\mathbb{U}\to\mathbb{U}$ is a linear operator, then the spectrum $\sigma(\mathbb{S})$ is convex compact and lies in the open ball $\mathbb{A}_{\mathbb{C}}(\alpha)\leq\mathcal{N}_{CB(\mathbb{U},\mathbb{U})}(\mathbb{S})$. Hence, $\rho(\mathbb{S})\neq\emptyset$.*

Proof.
Let $\alpha\neq 0$ and $\beta=\frac{1}{\alpha}$. Then, from Theorem 3.12, we have the representation $\Re_\alpha=(\mathbb{S}-\alpha\mathbb{I})^{-1}=-\frac{1}{\alpha}(\mathbb{I}-\beta\mathbb{S})^{-1}=-\frac{1}{\alpha}\sum_{j=0}^{\infty}(\beta\mathbb{S})^j=-\frac{1}{\alpha}\sum_{j=0}^{\infty}\left(\frac{1}{\alpha}\mathbb{S}\right)^j$, where, by Theorem 3.11, the infinite series converges for all α such that $\mathcal{N}_{CB(\mathbb{U},\mathbb{U})}\left(\frac{1}{\alpha}\mathbb{S}\right)=\frac{\mathcal{N}_{CB(\mathbb{U},\mathbb{U})}(\mathbb{S})}{\mathbb{A}_{\mathbb{C}}(\alpha)}<1$; that is, $\mathcal{N}_{CB(\mathbb{U},\mathbb{U})}(\mathbb{S})<\mathbb{A}_{\mathbb{C}}(\alpha)$. The same Theorem also shows that any $\alpha\in\rho(\mathbb{S})$. Hence, $\sigma(\mathbb{S})$ must lie in the convex open ball(6) so that $\sigma(\mathbb{S})$ is CB. Moreover, $\sigma(\mathbb{S})$ is convex closed by Theorem 3.12. Consequently, $\sigma(\mathbb{S})$ is convex compact.

Definition 3.14. *Let $(\mathbb{U},\mathcal{N})$ be convex complete c-NS and let $\mathbb{S}\in CB(\mathbb{U},\mathbb{U})$. Then, the spectrum radius $r_{\sigma(\mathbb{S})}(\mathbb{S})$ of an operator $\mathbb{S}$ is defined by $r_{\sigma(\mathbb{S})}(\mathbb{S})=\sup_{\alpha\in\sigma(\mathbb{S})}\mathbb{A}_{\mathbb{C}}(\alpha)$.*

Remark 3.15. *From (6) we see that* $r_{\sigma(\mathbb{S})}(\mathbb{S}) \leq \mathcal{N}_{CB(\mathbb{U},\mathbb{U})}(\mathbb{S})$.

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