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Reg-N-Flat Modules

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Abstract

Let R represent a ring. As a suitable generalization of N-flat (resp. flat) module, we present and investigate the concept of Reg-N-flat (resp. Reg-flat) right R-module. We give many properties and characterizations of these modules.

1 Introduction

In this paper, R represents an associative ring that has 1 and any module is unitary. Mod-R (resp. R-Mod) is the symbol for the right (resp. left) R-module category. If $M \in R$-Mod and L is a submodule of M with $0 \to D \otimes L \to D \otimes M$ is exact for all $D \in$Mod-R, then L is called pure in M [1]. If every submodule of a module $N \in R$-Mod is pure, then N is called regular [2]. The notation $N \leq^{reg} M$ (resp., $N \leq^{fgreg} M$) means N is a regular (resp., a finitely generated regular) submodule of M. Reg$(M) = \sum\{N : N \leq^{reg} M\}$. As usual, $M^* =$Hom$_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. In the literature, there are numerous generalizations of injectivity and module flatness provided [3], [4], [5], [6], [7], [8], [9], [10], [11], [12] and [13].
In this article, we present and examine Reg-N-flatness (resp. Reg-flatness)

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of modules which are a proper generalization of N-flatness (resp. flatness) of modules. Let N be in R-Mod and C be in Mod-R. C is then said to be Reg-N-flat if, for any $A \leq^{reg} N$, we get the exactness of $0 \to C \otimes_R A \xrightarrow{I_C \otimes i_A} C \otimes_R N$. A module C is said to as Reg-flat if it is Reg-R-flat. If all left R-homomorphisms from any $A \leq^{reg} B$ into C extend to B, then we say that a module C is Reg-B-injective (with B in R-Mod). When C is Reg-R-injective, it is referred to as Reg-injective. The notation (Reg-N-$\mathbb{F})_R$ (resp. (Reg-$\mathbb{F})_R$, $_R$(Reg-N-$\mathbb{I}$), $_R$(Reg-$\mathbb{I}$)) means the class of Reg-N-flat right (resp. Reg-flat right, Reg-N-injective left, Reg-injective left) modules. Several properties of Reg-N-flat right (resp. Reg-flat right) modules are given. We prove (Reg-N-$\mathbb{F})_R$ and (Reg-$\mathbb{F})_R$ are closed classes under isomorphisms, direct limits, direct sums, and direct summands. A version of Jinzhong Theorem [14, Theorem 1.1] for Reg-flat modules is given; that is, if a ring R is commutative and a module L is simple, we demonstrate that $L \in$ Reg-$\mathbb{F}$ iff $L \in$ Reg-$\mathbb{I}$. We give several characterizations of Reg-N-flat modules; for example, we show that D is an Reg-N-flat module if and only if the exactness of $0 \to D \otimes_R K \xrightarrow{I_D \otimes i_K} D \otimes N$ holds for any $K \leq^{reg} N$ with K being finitely generated.

2 Reg-N-Flat Modules

Definition 2.1. *Let $N \in R$-Mod and $M \in$ Mod-R. We say that M is Reg-N-flat if, for any $K \leq^{reg} N$, the sequence $0 \to M \otimes_R K \xrightarrow{I_M \otimes i_K} M \otimes N$ is exact, where $i_K : K \to N$ and $I_M : M \to M$ are the inclusion homomorphism and the identity homomorphism, respectively. A module M is referred to as Reg-flat if it is Reg-R-flat.*

Examples 2.2. ***(1)*** *Let $N \in R$-Mod. If Reg$(N) = 0$, then all right R-modules are Reg-N-flat.*
(2) *Let $N \in R$-Mod. Clearly, every N-flat (resp. flat) right R-module is Reg-N-flat (resp. reg-flat). In general, the converse is not true. Since $< 0 >$ is the only regular ideal of $\mathbb{Z}$, we have Reg$(\mathbb{Z}_{\mathbb{Z}}) = 0$. By (1) above, all $\mathbb{Z}$-modules are Reg-$\mathbb{Z}$-flat (= Reg-flat). So, $\mathbb{Z}_n$ is Reg-$\mathbb{Z}$-flat (Reg-flat) but it is not $\mathbb{Z}$-flat (= flat), for $n \geq 2$.*

Theorem 2.3. *Let $L \in R$-Mod and $N \in$ Mod-R. Then $N \in$ (Reg-L-$\mathbb{F})_R$ iff $N^* \in {}_R$(Reg-L-$\mathbb{I}$).*

Proof. Straightforward. □

If $\mathcal{G} \subseteq R$-Mod and $\mathcal{F} \subseteq$ Mod-R, then the pair $(\mathcal{F}, \mathcal{G})$ is referred to as almost dual if $\mathcal{G}$ is closed under direct products and summands and for any $M \in$ Mod-R, $M \in \mathcal{F} \Leftrightarrow M^* \in \mathcal{G}$ [3].

Proposition 2.4. $((\textit{Reg-}N\text{-}\mathbb{F})_R, {}_R(\textit{Reg-}N\text{-}\mathbb{I}))$ *is an almost dual pair.*

Proof. Since ${}_R(\text{Reg-}N\text{-}\mathbb{I})$ is a closed class under direct products and summands, the result follows from Theorem 2.3. □

The proof of the next corollary follows directly from [15, Proposition 4.2.8(1,3)] and Proposition 2.4.

Corollary 2.5. *The class* $(\textit{Reg-}N\text{-}\mathbb{F})_R$ *is closed under pure extensions, pure submodules, pure homomorphic images, direct limits, and direct sums.*

Proposition 2.6. *Let* $M \in$ Mod-R *and* $N \in R$-Mod. *Then* M *is Reg-*N*-flat iff the sequence* $0 \to M \otimes_R K \xrightarrow{I_M \otimes i_K} M \otimes N$ *is exact for any* $K \leq^{fgreg} N$.

Proof. $(\Rightarrow)$. This is clear.
$(\Leftarrow)$. Let $K \leq^{reg} N$. Let $\sum_{i=1}^{n} m_i \otimes k_i \in \ker(I_M \otimes i_K)$. Since $\sum_{i=1}^{n} m_i \otimes k_i \in M \otimes_R K$ with $(I_M \otimes i_K)(\sum_{i=1}^{n} m_i \otimes k_i) = 0$ in $M \otimes_R N$, we have $\sum_{i=1}^{n} m_i \otimes k_i = 0$ in $M \otimes_R N$. Let $K' =< k_1, k_2, \cdots, k_n >$. Since $K \leq^{reg} N$, we have $K' \leq^{reg} N$ (by [2, Theorem 6]). Thus $0 \to M \otimes_R K' \xrightarrow{I_M \otimes i_{K'}} M \otimes N$ is an exact sequence (by hypothesis) and hence $\ker(I_M \otimes i_{K'}) = 0$ in $M \otimes_R K'$. Since $\sum_{i=1}^{n} m_i \otimes k_i \in M \otimes_R K'$, we have $\sum_{i=1}^{n} m_i \otimes k_i \in \ker(I_M \otimes i_{K'})$. Thus $\sum_{i=1}^{n} m_i \otimes k_i = 0$ in $M \otimes_R K'$. Since $M \otimes_R K' \subseteq M \otimes_R K$, it follows that $\sum_{i=1}^{n} m_i \otimes k_i = 0$ in $M \otimes_R K$. Thus $\ker(I_M \otimes i_K) = 0$ and hence $I_M \otimes i_K$ is a monomorphism. Thus, M is a Reg-N-flat right R-module. □

Corollary 2.7. *Let* $M \in$ Mod-R. *Then* M *is Reg-flat iff the sequence* $0 \to M \otimes_R L \xrightarrow{I_M \otimes i_L} M \otimes_R R$ *is exact, for every* $L \leq^{fgreg} {}_R R$.

Theorem 2.8. *Let* $N \in R$-Mod *and* $M \in$ Mod-R *and consider the next statements:*
(1) $M \in (\textit{Reg-}N\text{-}\mathbb{F})_R$.
(2) $\mathrm{Tor}_1(M, N/K) = 0$, *for any* $K \leq^{reg} N$.
(3) $\mathrm{Tor}_1(M, N/K) = 0$, *for any* $K \leq^{fgreg} N$.

Then (2) $\Rightarrow$ (3) $\Rightarrow$ (1). *Moreover, when* N *is flat, then the three statements are equivalent.*

Proof. (2) $\Rightarrow$ (3). This is obvious.
(3) $\Rightarrow$ (1). Let $L \leq^{fgreg} N$. Thus the sequence $\text{Tor}_1(M, N/L) \to M \otimes_R L \to M \otimes_R N$ is exact, by [16, Theorem X.II.5.4(4)]. Since $\text{Tor}_1(M, N/L) = 0$, we have the exactness of $0 \to M \otimes_R L \to M \otimes_R N$ for any $L \leq^{fgreg} N$. By Proposition 2.6, M is Reg-N-flat.
(1) $\Rightarrow$ (2). Let $K \leq^{reg} N$. By hypotheses, the sequence $0 \to M \otimes_R K \to M \otimes_R N$ is exact. By [16, Theorem X.II.5.4(4)], the sequence $\text{Tor}_1(M, N) \to \text{Tor}_1(M, N/K) \to M \otimes_R K \to M \otimes_R N$ is exact. Since N is flat, we get from [16, Theorem X.II.5.4(2)] that $\text{Tor}_1(M, N) = 0$. Since $0 \to M \otimes_R K \to M \otimes_R N$ is exact, we have $\text{Tor}_1(M, N/K) = 0$. $\square$

Corollary 2.9. *Let $M \in$ Mod-R. Then the next statements are equivalent:*
(1) *M is Reg-flat.*
(2) $\text{Tor}_1(M, R/I) = 0$*, for each $I \leq^{reg} {}_R R$.*
(3) $\text{Tor}_1(M, R/I) = 0$*, for each $I \leq^{fgreg} {}_R R$.*

Proof. Take $N = {}_R R$ and apply Theorem 2.8. $\square$

The following lemma is easy to prove.

Lemma 2.10. *Let $M, N \in R$-Mod. If $\text{Ext}^1(N/L, M) = 0$ for all $L \leq^{reg} N$, then M is Reg-N-injective.*

Proposition 2.11. *Consider the following conditions for a commutative ring R and $M, N \in R$-Mod.*
(1) *M is Reg-N-flat.*
(2) $\text{Hom}_R(M, K) \in {}_R(\textit{Reg-}N\text{-}\mathbb{I})$*, for any injective module K.*
(3) *$M \otimes_R K \in (\textit{Reg-}N\text{-}\mathbb{F})_R$, for any flat module K.*
Then (2) $\Rightarrow$ (3) $\Rightarrow$ (1). *Moreover, if N is a flat module, then* (1) $\Rightarrow$ (2).

Proof. (2) $\Rightarrow$ (3). Since K is a flat module, K^* is injective. Thus Theorem 2.75 in [17, p.92] implies that $(M \otimes_R K)^* \cong \text{Hom}_R(M, K^*)$. By (2), $(M \otimes_R K)^* \in {}_R(\text{Reg-}N\text{-}\mathbb{I})$ and so $M \otimes_R K \in (\text{Reg-}N\text{-}\mathbb{F})_R$ (by Theorem 2.3).
(3) $\Rightarrow$ (1). By flatness of $K = {}_R R$, $M \otimes_R R$ is a Reg-N-flat module. Since $M \cong M \otimes_R R$, we have M is Reg-N-flat.
(1) $\Rightarrow$ (2). Let N (resp. K) be a flat (resp. an injective) module and let $L \leq^{reg} N$, thus $\text{Ext}^1(N/L, \text{Hom}_R(M, K)) \cong \text{Hom}_R(\text{Tor}_1(M, N/L), K)$. Since M is a Reg-N-flat module and N is a flat module (by hypothesis), $\text{Tor}_1(M, N/L) = 0$ (by Theorem 2.8) and hence $\text{Hom}_R(\text{Tor}_1(M, N/L), K) = 0$. Thus $\text{Ext}^1(N/L, \text{Hom}_R(M, K)) = 0$ and so from Lemma 2.10 we get $\text{Hom}_R(M, K)$ is a Reg-N-injective module. $\square$

Proposition 2.12. *Let M be a module with $ML = 0$ for any regular proper ideal L of a commutative ring R. If $M \in {}_R$(Reg-F), then* End(M) *is Reg-injective as R-module.*

Proof. Let $L \leq^{reg} {}_R R$ with $L \neq R$ and a module M be Reg-flat. By hypothesis, $ML = 0$. Since M is Reg-flat, we have $M \otimes_R L \cong ML = 0$. By [17, Theorem 2.76, p.93], $0 = \mathrm{Hom}_R(M \otimes_R L, M) \cong \mathrm{Hom}_R(L, \mathrm{End}(M))$. By [16, Theorem XII.4.4(3), p.491], the sequence $0 = \mathrm{Hom}_R(L, \mathrm{End}(M)) \to \mathrm{Ext}^1(R/L, \mathrm{End}(M)) \to \mathrm{Ext}^1(R, \mathrm{End}(M)) = 0$ is exact and hence $\mathrm{Ext}^1(R/L, \mathrm{End}(M)) = 0$. Thus End($M$) is a Reg-injective as R-module, by Lemma 2.10. □

Assume that the ring R is commutative. End(M) is a Reg-injective as an R-module if M is a Reg-flat semisimple module.

Corollary 2.13. *Let R be a commutative ring. End(M) is a Reg-injective as an R-module if M is a Reg-flat semisimple module.*

Proof. Use [18, Theorem 9.2.1, p. 218] and Proposition 2.12. □

The following theorem is a version of Jinzhong Theorem [14, Theorem 1.1] for Reg-flat modules.

Theorem 2.14. *If M is a simple module over a ring R that is commutative. Then $M \in (\text{Reg-}\mathbb{F})_R$ iff $M \in {}_R(\text{Reg-}\mathbb{I})$.*

Proof. Use Corollary 2.13, [19, Corollary 18.19, p.212] and [19, Proposition 18.14, p. 210]. □

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