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Integral Aspects of the Generalized Pell and Pell-Lucas Numbers

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Abstract

In this paper, we propose integral representations of the one-parameter k-Pell and k-Pell-Lucas numbers. Our results are also deduced with the Pell and Pell-Lucas numbers.

1 Introduction

Let P_n be the *Pell number* defined by $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$ with $P_0 = 0$ and $P_1 = 1$, and let Q_n be the *Pell-Lucas number* defined by $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$ with $Q_0 = 2$ and $Q_1 = 2$. In 2019, Trojnar-Spenlina [1] introduced one-parameter generalizations of the Pell and Pell-Lucas numbers by studying the recursive and defined them as follows:
Let k and n be non-negative integers with $k \geq 2$. A *one-parameter k-Pell*

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number $\mathcal{P}_{k,n}$ is defined by $\mathcal{P}_{k,n} = k\mathcal{P}_{k,n-1} + (k-1)\mathcal{P}_{k,n-2}$ for $n \geq 2$ with $\mathcal{P}_{k,0} = 0$ and $\mathcal{P}_{k,1} = 1$, and a *one-parameter k-Pell-Lucas number* $\mathcal{Q}_{k,n}$ is defined by $\mathcal{Q}_{k,n} = k\mathcal{Q}_{k,n-1} + (k-1)\mathcal{Q}_{k,n-2}$ for $n \geq 2$ with $\mathcal{Q}_{k,0} = 2$ and $\mathcal{Q}_{k,1} = 2$. For $k = 2$, the classical Pell and Pell–Lucas numbers are obtained. Binet's formulas for $\mathcal{P}_{k,n}$ and $\mathcal{Q}_{k,n}$ are

$$\mathcal{P}_{k,n} = \frac{1}{\Delta_k}\Big(\sigma_k^n - \frac{(1-k)^n}{\sigma_k^n}\Big) \tag{1.1}$$

and

$$\mathcal{Q}_{k,n} = \Big(1 - \frac{k-2}{\Delta_k}\Big)\sigma_k^n + \Big(1 + \frac{k-2}{\Delta_k}\Big)\frac{(1-k)^n}{\sigma_k^n}, \tag{1.2}$$

where $\Delta_k := \sqrt{k^2 + 4k - 4}$ and $\sigma_k := \frac{1}{2}(k + \Delta_k)$, see [1, Corollary 2.3].

In 2024, Nilsrakoo [2] presented simple integral representations for P_n and Q_n by using a technique in [3]. In this paper, we propose new integral representations of the one-parameter k-Pell and k-Pell-Lucas numbers.

2 Main results

First, we give some identities by using Binet's formulas (1.1) and (1.2).

Lemma 2.1. *Let k and n be non-negative integers with $k \geq 2$. Then*

(i) $\mathcal{Q}_{k,n} + (k - 2 + \Delta_k)\, \mathcal{P}_{k,n} = 2\sigma_k^n$;

(ii) $\mathcal{Q}_{k,n} + (k - 2 - \Delta_k)\, \mathcal{P}_{k,n} = 2\frac{(1-k)^n}{\sigma_k^n}$;

(iii) $(\mathcal{Q}_{k,n} + (k-2)\mathcal{P}_{k,n})^2 - \Delta_k^2\mathcal{P}_{k,n}^2 = 4(1-k)^n$.

Lemma 2.2. *Let k, m and n be non-negative integers with $k \geq 2$. Then*

(i) $2\mathcal{P}_{k,m+n} = \mathcal{P}_{k,m}\mathcal{Q}_{k,n} + \mathcal{P}_{k,n}\mathcal{Q}_{k,m} + 2(k-2)\mathcal{P}_{k,m}\mathcal{P}_{k,n}$;

(ii) $2\mathcal{Q}_{k,m+n} = \mathcal{Q}_{k,m}\mathcal{Q}_{k,n} + 8(k-1)\mathcal{P}_{k,m}\mathcal{P}_{k,n}$.

Now, we obtain a new integral representation for $\mathcal{P}_{k,\ell n}$ can be found by employing other known relations between the two numbers $\mathcal{P}_{k,\ell}$ and $\mathcal{Q}_{k,\ell}$.

Theorem 2.3. *Let k, ℓ and n be non-negative integers with $k \geq 2$. Then*

$$\mathcal{P}_{k,\ell n} = \frac{n\mathcal{P}_{k,\ell}}{2^n\Delta_k}\int_{-\Delta_k}^{\Delta_k} \left(\mathcal{Q}_{k,\ell} + (k-2+x)\,\mathcal{P}_{k,\ell}\right)^{n-1} dx. \tag{2.3}$$

Proof. A simple integration leads to

$$\int_{-\Delta_k}^{\Delta_k} (\mathcal{Q}_{k,\ell} + (k-2+x)\,\mathcal{P}_{k,\ell})^{n-1}\,dx = \frac{1}{n\,\mathcal{P}_{k,\ell}}\Big[\,(\mathcal{Q}_{k,\ell} + (k-2+x)\,\mathcal{P}_{k,\ell})^{n}\,\Big]_{-\Delta_k}^{\Delta_k}$$
$$= \frac{1}{n\mathcal{P}_{k,\ell}}\left[(\mathcal{Q}_{k,\ell} + (k-2+\Delta_k)\,\mathcal{P}_{k,\ell})^{n}\right] - \frac{1}{n\mathcal{P}_{k,\ell}}\left[(\mathcal{Q}_{k,\ell} + (k-2-\Delta_k)\,\mathcal{P}_{k,\ell})^{n}\right].$$

From (i) and (ii) of Lemma 2.1 with n replaced with ℓ, it follows that

$$\begin{aligned}\int_{-\Delta_k}^{\Delta_k} (\mathcal{Q}_{k,\ell} + (k-2+x)\,\mathcal{P}_{k,\ell})^{n-1}\,dx &= \frac{1}{n\mathcal{P}_{k,\ell}}\left[\left(2\sigma_k^{\ell}\right)^n - \left(2\frac{(1-k)^{\ell}}{\sigma_k^{\ell}}\right)^n\right]\\ &= \frac{2^n\Delta_k}{n\mathcal{P}_{k,\ell}}\left[\frac{1}{\Delta_k}\left(\sigma_k^{\ell n} - \frac{(1-k)^{\ell n}}{\sigma_k^{\ell n}}\right)\right]\\ &= \frac{2^n\Delta_k\,\mathcal{P}_{k,\ell n}}{n\mathcal{P}_{k,\ell}}.\end{aligned}$$

Then (2.3) follows. This completes the proof. □

Remark 2.4. *As in Theorem 2.3, equation* (2.3) *is equivalent to*

$$\mathcal{P}_{k,\ell n} = \frac{n\mathcal{P}_{k,\ell}}{2^n}\int_{-1}^{1} (\mathcal{Q}_{k,\ell} + (k-2+\Delta_k\,t)\,\mathcal{P}_{k,\ell})^{n-1}\,dt.$$

Indeed, substituting $t = \frac{x}{\Delta_k}$ *produces* $dx = \Delta_k dt$ *and the limits of integration are changed to* -1 *and* 1*, respectively.*

Since $\mathcal{P}_{2,n} = P_n$, we get the following corollary.

Corollary 2.5 ([2], Theorem 3.1). *Let* ℓ *and* n *be non-negative integers. Then*

$$P_{\ell n} = \frac{nP_\ell}{2^n\sqrt{8}}\int_{-\sqrt{8}}^{\sqrt{8}} (Q_\ell + P_\ell\,x)^{n-1}dx = \frac{nP_\ell}{2^n}\int_{-1}^{1} (Q_\ell + \sqrt{8}\,P_\ell\,t)^{n-1}dt.$$

Finally, we provide the integral representations for $\mathcal{Q}_{k,\ell n}$ based on the two numbers $\mathcal{P}_{k,\ell}$ and $\mathcal{Q}_{k,\ell}$.

Theorem 2.6. *Let* k*,* ℓ *and* n *be non-negative integers with* $k \geq 2$*. Then* $\mathcal{Q}_{k,\ell n}$ *is equal to*

$$\frac{1}{2^n\Delta_k}\int_{-\Delta_k}^{\Delta_k} (\mathcal{Q}_{k,\ell} + (k-2+x-n(k-2-x))\mathcal{P}_{k,\ell})(\mathcal{Q}_{k,\ell} + (k-2+x)\mathcal{P}_{k,\ell})^{n-1}dx.$$

Proof. Replacing n by $n+1$ in (2.3), we get

$$\mathcal{P}_{k,\ell n+\ell} = \frac{(n+1)\mathcal{P}_{k,\ell}}{2^{n+1}\Delta_k}\int_{-\Delta_k}^{\Delta_k} \left(\mathcal{Q}_{k,\ell} + (k-2+x)\,\mathcal{P}_{k,\ell}\right)^n dx. \tag{2.4}$$

Using the integration by parts with (2.4), Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned}
I &= \frac{1}{2^n\Delta_k}\int_{-\Delta_k}^{\Delta_k} \left(\mathcal{Q}_{k,\ell} + (n+1)\,(k-2+x)\,\mathcal{P}_{k,\ell}\right)\left(\mathcal{Q}_{k,\ell} + (k-2+x)\,\mathcal{P}_{k,\ell}\right)^{n-1} dx \\
&= \frac{1}{n2^n\Delta_k\,\mathcal{P}_{k,\ell}}\left[\left(\mathcal{Q}_{k,\ell} + (k-2+\Delta_k)\,\mathcal{P}_{k,\ell}\right)^n\left(\mathcal{Q}_{k,\ell} + (n+1)\,(k-2+\Delta_k)\,\mathcal{P}_{k,\ell}\right)\right] \\
&\quad - \frac{1}{n2^n\Delta_k\,\mathcal{P}_{k,\ell}}\left[\left(\mathcal{Q}_{k,\ell} + (k-2-\Delta_k)\,\mathcal{P}_{k,\ell}\right)^n\left(\mathcal{Q}_{k,\ell} + (n+1)\,(k-2-\Delta_k)\,\mathcal{P}_{k,\ell}\right)\right] \\
&\quad - \frac{2\mathcal{P}_{k,\ell n+\ell}}{n\mathcal{P}_{k,\ell}} \\
&= \frac{1}{n2^n\Delta_k\,\mathcal{P}_{k,\ell}}\left[2^n\sigma_k^{\ell n}\left(\mathcal{Q}_{k,\ell} + (n+1)\,(k-2+\Delta_k)\,\mathcal{P}_{k,\ell}\right)\right] \\
&\quad - \frac{1}{n2^n\Delta_k\,\mathcal{P}_{k,\ell}}\left[2^n\frac{(1-k)^{\ell n}}{\sigma_k^{\ell n}}\left(\mathcal{Q}_{k,\ell} + (n+1)\,(k-2-\Delta_k)\,\mathcal{P}_{k,\ell}\right)\right] - \frac{2\mathcal{P}_{k,\ell n+\ell}}{n\mathcal{P}_{k,\ell}} \\
&= \frac{1}{n\mathcal{P}_{k,\ell}}\left[\frac{1}{\Delta_k}\left(\sigma_k^{\ell n} - \frac{(1-k)^{\ell n}}{\sigma_k^{\ell n}}\right)\right]\left[\mathcal{Q}_{k,\ell} + (n+1)\,(k-2)\,\mathcal{P}_{k,\ell}\right] \\
&\quad + \frac{1}{n\mathcal{P}_{k,\ell}}\left(\sigma_k^{\ell n} + \frac{(1-k)^{\ell n}}{\sigma_k^{\ell n}}\right)(n+1)\,\mathcal{P}_{k,\ell} - \frac{2\mathcal{P}_{k,\ell n+\ell}}{n\mathcal{P}_{k,\ell}} \\
&= \frac{1}{n\mathcal{P}_{k,\ell}}\mathcal{P}_{k,\ell n}\left[\mathcal{Q}_{k,\ell} + (n+1)\,(k-2)\,\mathcal{P}_{k,\ell}\right] \\
&\quad + \frac{1}{n\mathcal{P}_{k,\ell}}\left(\mathcal{Q}_{k,\ell n} + (k-2)\mathcal{P}_{k,\ell n}\right)(n+1)\,\mathcal{P}_{k,\ell} - \frac{2\mathcal{P}_{k,\ell n+\ell}}{n\mathcal{P}_{k,\ell}} \\
&= \mathcal{Q}_{k,\ell n} + 2\,(k-2)\,\mathcal{P}_{k,\ell n}.
\end{aligned}$$

Then $\mathcal{Q}_{k,\ell n} = I - 2\,(k-2)\,\mathcal{P}_{k,\ell n}$. Applying Theorem 2.3, the proof is complete. □

Remark 2.7. *Setting $k = 2$ in Theorem 2.6, we obtain [2, Theorem 3.4].*

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