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# Generalized Hamiltonian Transformations

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**Abstract**

In this article, we discuss the $G$-Hamiltonian system which is a generalization of the traditional Hamiltonian system in mechanics. Moreover, we demonstrate that when given a $G$-Hamiltonian system of differential equations, such as $\frac{dx}{dt} = Ax$, the transformations that have this form $\frac{dy}{dt} = BA^{-1}By$ are also $G$-Hamiltonian. Furthermore, we discuss the $G$-Lagrangian structure which is a generalization of the traditional Lagrangian formulation in mechanics.

## 1 Introduction

In Mathematics, a Hamiltonian System is a system of differential equations which can be written in the form of Hamilton's equations. Hamiltonian Systems are usually formulated in terms of Hamiltonian vector fields on symplectic manifolds or Poisson manifolds. Many problems in engineering and physics lead to investigation of Hamiltonian systems of differential equations with periodic coefficients [4, 5]. As described in many books such as [1, 4, 5, 6, 7], a classical Hamiltonian system is typically represented by the

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following set of differential equations:

$$\frac{dx}{dt} = J\delta H(x),$$

where $x$ represents the state vector of the system (e.g., position and momentum), $H(x)$ is the Hamiltonian, a scalar function representing the total energy of the system which is typically the sum of the kinetic and potential energies, and $J$ is a symplectic matrix, usually a block matrix of the form

$$\mathbf{J} = \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}. \tag{1.1}$$

The $G$-Hamiltonian system is a generalization of the traditional Hamiltonian system. A traditional Hamiltonian system is described by differential equations of the form:

$$\frac{dx}{dt} = Ax,$$

where $x$ is a vector of state variables and $A$ is a matrix representing the system's dynamics. In this generalization, the system is modified or "transformed" by introducing a matrix $B$ and considering the structure of the system in a generalized Hamiltonian framework. The transformation leads to a new system:

$$\frac{dy}{dt} = BAB^{-1}y.$$

The system remains Hamiltonian, as suggested by the term "$G$-Hamiltonian" meaning that even after the transformation, the system retains certain key properties characteristic of Hamiltonian dynamics such as conserved quantities or symplectic structure [2]. The paper is organized as follows: In Section 2, we discuss some basic definitions and properties of the $G$-Hamiltonian system. In Section 3, we introduce the main result. In Section 4, we discuss the structure of the $G$-Lagrangian solutions.

## 2 Basic Definition and properties of $G$-Hamiltonian Systems

A natural way of studying Hamiltonian Systems is to linearize them about their periodic solutions and changing system coordinates to obtain new systems which are called $G$-Hamiltonian Systems. A $G$-Hamiltonian System was introduced in 1975 as a generalization of the well-known Hamiltonian system [3]. We are interested in studying the $G$-Hamiltonian System which has the form

$$\dot{x} = GA(t)x, \tag{2.2}$$

where the matrix $G$ is anti-symmetric and the matrix $A$ is Hermitian (i.e., $G^T = -G, A^* = A$). We can easily see that a $G$-Hamiltonian matrix function $M(t) = GA(t)$ is the coefficient matrix of a Hamiltonian equation in matrix notation $\dot{x} = M(t)x$. From now on, we consider the following definition:

**Definition 2.1.** *A matrix $A \in \mathbb{C}^{n\times n}$ is called G-Hamiltonian (or G-anti-Hermitian) if it satisfies the relation $A^G = -A$.*

Each $G$-Hamiltonian $A$ has the representation $A = i^{-1}GH$ with some $H = H^*$. For a G-Unitary matrix, we have $A^G = G^{-1}A^*G$, the set of all multipliers of $A$ is symmetric with respect to the unit circle; i.e., together with any its eigenvalue $\rho$ the matrix $A$ has the eigenvalue $1/\bar{\rho}$. Similarly, if $A$ is a $G$-Hamiltonian matrix, then $A = -G^{-1}A^*G$, and the set of all multipliers of $A$ is symmetric with respect to the imaginary axis; i.e., if $\lambda$ is an eigenvalue of $A$, then $-\bar{\lambda}$ is an eigenvalue of $A$ too.

**Remark 2.2.** *Under the transformation $x = By$, system* (1) *becomes $\dot{y} = JB^{-1}ABy$. When $x_1 = By_1$, $x_2 = By_2$, the form $(Gx_1, x_2)$ becomes $(GBy_1, By_2) = (B^*GBy_1, y_2)$ and the new form for $G$ will denoted by $G' = B^*GB$.*

# 3 The Main Result

**Theorem 3.1.** *If the system $\dot{x} = Ax$ is G-Hamiltonian, then, under the transformation $x = By$, the new system $\dot{y} = B^{-1}ABy$ is also a G-Hamiltonian system.*

*Proof.* Since the new $G$ form is given by $G' = B^*GB$, by assumption we have $A^G = -A$, where $A^G$ is given in section (2). Let $M = B^{-1}AB$. The goal is to prove that $M^{G'} = -M$.

$$\begin{aligned}
M^{G'} &= {G'}^{-1}M^*G' \\
&= (B^*GB)^{-1}(B^{-1}AB)^*(B^*GB) \\
&= B^{-1}G^{-1}(B^*)^{-1}B^*A^*(B^*)^{-1}B^*GB \\
&= B^{-1}G^{-1}A^*GB \\
&= B^{-1}A^GB, \qquad (A^G = G^{-1}A^*G = -A) \\
&= -B^{-1}AB \\
&= -M.
\end{aligned}$$

□

**Theorem 3.2.** *The system $\dot{x} = GA(t)x$ is G-Hamiltonian under the following conditions:*
*1) The matrix A is Hermitian (i.e., $A^* = A$)*
*2) The matrix G is non-singular and anti-Hermitian matrix (i.e., $\det G \neq 0$, $G^* = -G$).*

*Proof.* Let $M = GA(t)$. Set $G' = A^*GA$. To prove the system is $G$-Hamiltonian, we must show that $M^{G'} = -M$, where $M^{G'} = {G'}^{-1}M^*G'$,

$$\begin{aligned} M^{G'} = (GA)^{G'} &= {G'}^{-1}(GA)^*G' \\ &= (A^*GA)^{-1}(GA)^*A^*GA \\ &= A^{-1}G^{-1}(A^*)^{-1}A^*G^*A^*GA \\ &= A^{-1}G^{-1}G^*A^*GA \\ &= -GA, \qquad (A^* = A, G^* = -G) \\ &= -M. \end{aligned}$$

□

# 4 Structure of the $G$-Lagrangian Solutions

**Definition 4.1.** *For two smooth vector functions $X_1, X_2 : I \to \mathbb{R}^{2n}$, we define the Poisson bracket of $X_1, X_2$ to be $\{X_1, X_2\}(t) = X_1^T(t)GX_2(t)$.*

**Definition 4.2.** *$X_1, X_2$ are said to be G-involution if $\{X_1, X_2\} \equiv 0$*

**Definition 4.3.** *A set of n linearly independent functions $X_1, .....X_n$ which are pairwise G-involution functions are said to be a G-Lagrangian set.*

We now state and prove the following theorem:

**Theorem 4.4.** *If a G-Lagrangian set of solutions of the equation $\dot{x} = GA(t)x$ is given, then a complete set of $2n$ linearly independent solutions can be found by integration.*

*Proof.* We have $\dot{x} = GA(t)x$, where $A(t)$ is symmetric, $G$ is anti-Symmetric and non-singular matrix. Let $R(t)$ be the $2n \times n$ matrix whose columns are the $n$-linearly independent solutions. If $R(t)$ is a $G$-Lagrangian solutions, then we can construct the full fundamental matrix solution, since the columns are solutions so that $\dot{R} = GAR$ and they are G-involution (i.e., $R^T GR = 0$). As a special case assume that $G^2 = -CI$, where $C$ means constant. We define a $2n \times n$ matrix by $D_{2n\times n} = GR(R^T R)^{-1}$. Then

$$D^T GD = 0, \qquad R^T GD = -CI, \qquad (C = Constant).$$

If $K = (D, R)$, then $K$ must satisfy the following condition: $K^T GK = CJ$, and hence

$$K^{-1} = \frac{1}{C} J^{-1} K^T G = \begin{pmatrix} 0 & -\frac{1}{C} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} D^T G \\ R^T G \end{pmatrix} = \frac{1}{C} \begin{pmatrix} -R^T G \\ D^T G \end{pmatrix}.$$

Now, change coordinates by $x = K\zeta$ so that $\dot{x} = \dot{K}\zeta + K\dot{\zeta}$,

$$\begin{aligned} \dot{\zeta} &= K^{-1}(GAK - \dot{K})\zeta \\ &= \frac{1}{C}\begin{pmatrix} -R^T G \\ D^T G \end{pmatrix}(GAD - \dot{D}, GAR - \dot{R})\zeta \\ &= \frac{1}{C}\begin{pmatrix} -R^T G^2 AD + R^T G\dot{D} & -R^T G^2 AR + R^T G\dot{R} \\ D^T G^2 AD - D^T G\dot{D} & D^T G^2 AR - D^T G\dot{R} \end{pmatrix}\zeta \\ &= \frac{1}{C}\begin{pmatrix} 0 & 0 \\ D^T G^2 AD - D^T G\dot{D} & 0 \end{pmatrix}\zeta \end{aligned}$$

Due to the fact that $\dot{R} = GAR$, the second column in the above matrix is zero. The one in the upper left-hand corner is also zero, which can be easily seen by differentiating the fact that $R^T GD = -CI$. Therefore, $\dot{v_1} = 0$, and $\dot{v_2} = \frac{D^T}{C}(G^2 AD - G\dot{D})v_1$,

$$\zeta = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

which has a general solution

$$\begin{aligned} v_1 &= v_{10}, \\ v_2 &= v_{20} + Lv_{10}, \end{aligned}$$

where

$$L = \frac{1}{C}\int_{t_0}^{t} D^T(G^2 AD + G\dot{D})dt.$$

Therefore, a symplectic fundamental matrix solution is

$$X = (D + RL, R).$$

As a result, the complete set of solutions is obtained by integration. □

# 5 Conclusion

The generalization of the Hamiltonian system to the $G$-Hamiltonian system provides a more flexible and robust framework for studying dynamical systems. By extending the classical Hamiltonian mechanics, the $G$-Hamiltonian system introduces transformations that preserve the core properties of the original Hamiltonian. In this paper, we showed that, given systems of $G$-Hamiltonian equations, the transformations are also $G$-Hamiltonian provided that the matrix inverse exists.

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