International Journal of Mathematics and
Computer Science, **20**(2025), no. 1, 79–88
DOI: https://doi.org/10.69793/ijmcs/01.2025/wiwatwanich

A New Double Integral Transform for Solving Partial Integro-Differential Equation

Duangkamol Poltem, Araya Wiwatwanich

Department of Mathematics
Faculty of Science
Burapha University
Chonburi, Thailand

email: arayawi@go.buu.ac.th

(Received June 7, 2024, Accepted July 14, 2024, Published July 15, 2024)

Abstract

In this paper, we propose a new double integral transform operator, called the Laplace-General transform. Moreover, we establish several basic properties and fundamental theorems of the Laplace-General transform. Furthermore, we apply The theoretical results to a class of partial integro-differential equations as an illustrative example.

1 Introduction

The Laplace transform has been extensively studied for decades as a powerful tool for solving various types of differential equations. Variations of the Laplace transformation, such as the double Laplace transform, have been developed to address problems involving two independent variables [1, 2]. Recently, the Laplace-Sumudu transform (LST) has also gained attention for its unique properties and applications [3]. In line with these advancements, it is reasonable to explore the combination of the Laplace transform with other integral transforms. In this work, we demonstrate how to solve the integral differential equations by combining the Laplace and the general integral transforms.

Key words and phrases: Integral transform, Partial integro-differential equations, Convolution.
AMS (MOS) Subject Classifications: 44A05, 44A35.
ISSN 1814-0432, 2025, http://ijmcs.future-in-tech.net

2 A new double integral transform

In this section, we give the basic concepts of the Laplace-General transform (LGT) adopted in the paper are given.

Definition 2.1. *The Lapace transform of the continuous function $f(x)$ is defined by:*

$$L[f(x)] = \mathcal{F}(\alpha) = \int_0^\infty e^{-\alpha x} f(x)dx. \tag{2.1}$$

Definition 2.2. *The general integral transform [4] of the continuous function $f(t)$ is*

$$T[f(t)] = \mathcal{F}(\beta) = p(\beta)\int_0^\infty e^{-q(\beta)t} f(t)dt. \tag{2.2}$$

Clearly, if $p(\beta) = 1$ and $q(\beta) = \alpha$, then this general transform (2.2) gives the Laplace transform (2.1).

Definition 2.3. *The Laplace-General transform (LGT) of the function $f(x,t)$ of two variables $x > 0$ and $t > 0$ is defined by*

$$L_x T_t[f(x,t)] = \mathcal{F}(\alpha,\beta) = p(\beta)\int_0^\infty \int_0^\infty e^{-\alpha x - q(\beta)t} f(x,t)dtdx. \tag{2.3}$$

Clearly, the LGT is a linear transformation. and covers most or even all type of double integral transforms in the family of Laplace transform for different value of $p(s)$ and $q(s)$ such as Laplace-Laplace [5], Laplace-Sumudu [3] transforms.

Definition 2.4. *A function $f(x,t)$ is said to be of exponential order c and d if there exists a positive constant K, $\mathcal{X} > 0$, and $\mathcal{T} > 0$ such that $\forall x > \mathcal{X}$, $\forall t > \mathcal{T}$,*

$$|f(x,t)| \le Ke^{cx+dt}, \tag{2.4}$$

or, equivalently,

$$\begin{aligned} \lim_{x\to\infty, t\to\infty} e^{-\alpha x - q(\beta)t}|f(x,t)| &= K \lim_{x\to\infty, t\to\infty} e^{(-\alpha-c)x-(q(\beta)-d)t}, \\ &= 0, \ \alpha > c, \, q(\beta) > d. \end{aligned} \tag{2.5}$$

The function $f(x,t)$ is called an exponential order as $x \to \infty$, $t \to \infty$ and clearly, it does not grow faster than Ke^{cx+dt} as $x \to \infty$, $t \to \infty$.

Theorem 2.5. *Let a function $f(x,t)$ be a continuous function in every finite interval $(0,\mathcal{X})$ and $(0,\mathcal{T})$ of exponential order e^{cx+dt}. Then the Laplace-General transform of $f(x,t)$ exists for all α and $q(\beta)$ with $Re(\alpha) > c$ and $Re(q(\beta)) > d$.*

Proof. We have

$$\begin{aligned} |\mathcal{F}(\alpha,\beta)| &= \left| p(\beta)\int_0^\infty\int_0^\infty e^{-\alpha x - q(\beta)t} f(x,t)\,dtdx \right| \\ &\leq Kp(\beta)\int_0^\infty\int_0^\infty e^{-\alpha x - q(\beta)t} e^{cx+dt}\,dtdx \qquad (2.6) \\ &= \frac{Kp^2(\beta)}{(\alpha - c)(q(\beta)-d)},\ Re(\alpha) > c,\ Re(q(\beta)) > d. \end{aligned}$$

It follows that

$$\lim_{x\to\infty, t\to\infty} |\mathcal{F}(\alpha,\beta)| = 0 \quad \text{or} \lim_{x\to\infty, t\to\infty} \mathcal{F}(\alpha,\beta) = 0.$$

□

3 The LGT of some basic functions

By Definition (2.3), we obtain the following useful properties:

(1) $L_xT_t[1] = \dfrac{p(\beta)}{\alpha q(\beta)}$.

(2) $L_xT_t[x^c t^d] = p(\beta)\dfrac{\Gamma(c+1)}{\alpha^{c+1}}\dfrac{\Gamma(d+1)}{[q(\beta)]^{d+1}},\ Re(c) > -1,\ Re(d) > -1.$

If c and d are positive integers, then $L_xT_t[x^c t^d] = p(\beta)\dfrac{c!d!}{\alpha^{c+1}[q(\beta)]^{d+1}}$.

(3) $L_xT_t[e^{cx+dt}] = \dfrac{p(\beta)}{(\alpha-c)(q(\beta)-d)}$.

Similarly,

$L_xT_t[e^{i(cx+dt)}] = \dfrac{p(\beta)}{(\alpha-ci)(q(\beta)-di)} = p(\beta)\left[\dfrac{(\alpha q(\beta) - cd) + i(cq(\beta) + d\alpha)}{(\alpha^2+c^2)(q^2(\beta)+d^2)}\right]$.

(4) $L_xT_t[\sin(cx+dt)] = p(\beta)\left[\dfrac{(cq(\beta)+d\alpha)}{(\alpha^2+c^2)(q^2(\beta)+d^2)}\right]$.

(5) $L_xT_t[\cos(cx+dt)] = p(\beta)\left[\dfrac{(\alpha q(\beta)-cd)}{(\alpha^2+c^2)(q^2(\beta)+d^2)}\right].$

(6) $L_xT_t[\sinh(cx+dt)] = p(\beta)\left[\dfrac{(cq(\beta)+d\alpha)}{(\alpha^2-c^2)(q^2(\beta)-d^2)}\right].$

(7) $L_xT_t[\cosh(cx+dt)] = p(\beta)\left[\dfrac{(\alpha q(\beta)-cd)}{(\alpha^2-c^2)(q^2(\beta)-d^2)}\right].$

(8) $L_xT_t[g(x)h(t)]) = L_x[g(x)]T_t[h(t)].$

Theorem 3.1. *Let $\mathcal{F}(\alpha,\beta) = L_xT_t[f(x,t)]$, then*

(i) $L_xT_t\left[\dfrac{\partial f(x,t)}{\partial x}\right] = \alpha\mathcal{F}(\alpha,\beta) - T[f(0,t)],$

(ii) $L_xT_t\left[\dfrac{\partial f(x,t)}{\partial t}\right] = q(\beta)\mathcal{F}(\alpha,\beta) - p(\beta)L[f(x,0)],$

(iii) $L_xT_t\left[\dfrac{\partial^2 f(x,t)}{\partial x^2}\right] = \alpha^2\mathcal{F}(\alpha,\beta) - \alpha T[f(0,t)] - T[f_x(0,t)],$

(iv) $L_xT_t\left[\dfrac{\partial^2 f(x,t)}{\partial t^2}\right] = q^2(\beta)\mathcal{F}(\alpha,\beta) - p^2(\beta)L[f(x,0)] - p(\beta)L[f_t(x,0)],$

(v) $L_xT_t\left[\dfrac{\partial^2 f(x,t)}{\partial x\partial t}\right] = \alpha q(\beta)\mathcal{F}(\alpha,\beta) - \alpha p(\beta)L[f(x,0)] - T[f_t(0,t)].$

Proof. (i)

$$\begin{aligned} L_xT_t\left[\frac{\partial f(x,t)}{\partial x}\right] = &= p(\beta)\int_0^\infty\int_0^\infty e^{-\alpha x - q(\beta)t}\frac{\partial f(x,t)}{\partial x}dtdx, \\ &= p(\beta)\int_0^\infty e^{-q(\beta)t}\left(\alpha\int_0^\infty e^{-\alpha x}f(x,t)dx - f(0,t)\right)dt, \\ &= \alpha F(\alpha,\beta) - T[f(0,t)]. \end{aligned}$$

(ii)

$$\begin{aligned} L_xT_t\left[\frac{\partial f(x,t)}{\partial t}\right] = &= p(\beta)\int_0^\infty\int_0^\infty e^{-\alpha x - q(\beta)t}\frac{\partial f(x,t)}{\partial t}dtdx, \\ &= p(\beta)\int_0^\infty e^{-\alpha x}\left(q(\beta)\int_0^\infty e^{-q(\beta)t}f(x,t)dt - f(x,0)\right)dx, \\ &= q(\beta)F(\alpha,\beta) - p(\beta)L[f(x,0)]. \end{aligned}$$

Similarly, we can prove the remaining properties. □

Theorem 3.2. *If $\mathcal{F}(\alpha,\beta) = L_xT_t[f(x,t)]$, then*

$$L_xT_t[f(x-\xi,t-\eta)H(x-\xi,t-\eta)] = e^{-\alpha\xi-q(\beta)\eta}\mathcal{F}(\alpha,\beta). \tag{3.7}$$

where $H(x,t)$ is the Heaviside unit step function defined by $H(x-\xi,t-\eta)=1$ when $x>\xi$ and $t>\eta$: $H(x-\xi,t-\eta)=0$ when $x<\xi$ and $t<\eta$.

Proof. By Definition (2.3), we have
$L_xT_t[f(x-\xi,t-\eta)H(x-\xi,t-\eta)]$

$$\begin{aligned} &= p(\beta)\int_0^\infty\int_0^\infty e^{-\alpha x-q(\beta)t}f(x-\xi,t-\eta)H(x-\xi,t-\eta)dtdx,\\ &= p(\beta)\int_\xi^\infty\int_\eta^\infty e^{-\alpha x-q(\beta)t}f(x-\xi,t-\eta)dtdx.\end{aligned}$$

Substituting $x-\xi=\tau$ and $t-\eta=\delta$, we ger

$$\begin{aligned} &= p(\beta)e^{-\alpha\xi-q(\beta)\eta}\int_0^\infty\int_0^\infty e^{-\alpha\tau-q(\beta)\delta}f(\tau,\delta)d\tau d\delta,\\ &= e^{-\alpha\xi-q(\beta)\eta}\mathcal{F}(\alpha,\beta).\end{aligned}$$

□

Definition 3.3. *The convolution of two integrable functions $f(x,t)$ and $g(x,t)$, denoted by $(f**g)(x,t)$, is defined by*

$$(f**g)(x,t) = \int_0^x\int_0^t f(x-\xi,t-\eta)g(\xi,\eta)d\xi d\eta. \tag{3.8}$$

Theorem 3.4. *(Convolution Theorem)*
Let $L_xT_t[f(x,t)] = \mathcal{F}(\alpha,\beta)$ and $L_xT_t[g(x,t)] = \mathcal{G}(\alpha,\beta)$ then

$$L_xT_t[(f**g)(x,t)] = \frac{1}{p(\beta)}\mathcal{F}(\alpha,\beta)\mathcal{G}(\alpha,\beta). \tag{3.9}$$

Proof. By Definition (2.3), we have
$L_xT_t[(f**g)(x,t)] = p(\beta)\int_0^\infty\int_0^\infty e^{-\alpha x-q(\beta)t}(f**g)(x,t)dtdx$,

$$= p(\beta)\int_0^\infty\int_0^\infty e^{-\alpha x-q(\beta)t}\left[\int_0^x\int_0^t f(x-\xi,t-\eta)g(\xi,\eta)\right]d\xi d\eta dtdx.$$

Using the Heaviside unit step function, we obtain

$$\begin{aligned} &= p(\beta)\int_0^\infty\int_0^\infty e^{-\alpha x-q(\beta)t}\left[\int_0^\infty\int_0^\infty f(x-\xi,t-\eta)H(x-\xi,t-\eta)g(\xi,\eta)\right]d\xi d\eta dt dx,\\ &= p(\beta)\int_0^\infty\int_0^\infty g(\xi,\eta)\left[\int_0^\infty\int_0^\infty e^{-\alpha x-q(\beta)t}f(x-\xi,t-\eta)H(x-\xi,t-\eta)\right]dt dx d\xi d\eta. \end{aligned}$$

By Theorem (3.2), we have

$$\begin{aligned} &= \int_0^\infty\int_0^\infty g(\xi,\eta)e^{-\alpha\xi-q(\beta)\eta}\mathcal{F}(\alpha,\beta)d\xi d\eta,\\ &= \mathcal{F}(\alpha,\beta)\int_0^\infty\int_0^\infty g(\xi,\eta)e^{-\alpha\xi-q(\beta)\eta}d\xi d\eta,\\ &= \frac{1}{p(\beta)}\mathcal{F}(\alpha,\beta)\mathcal{G}(\alpha,\beta). \end{aligned}$$

□

4 An Application to Class of Integro-Differential Equations

4.1 Voltera Integral Equation

Let us consider the Voltera integral equation in the form

$$\phi(x,t) = g(x,t) + \lambda\int_0^x\int_0^t \phi(x-\delta,t-\epsilon)\psi(\delta,\epsilon)d\delta d\epsilon, \tag{4.10}$$

where $\phi(x,t)$ is an unknown function, λ is a constant, and $g(x,t)$ and $\varphi(x,t)$ are two known functions.
Applying the Laplace-General transform (LGT) to both sides of Eq. (4.10) and using Theorem 3.4, we have

$$\overline{\phi}(\alpha,\beta) = \frac{p(\beta)\overline{g}(\alpha,\beta)}{p(\beta)-\lambda\overline{\psi}(\alpha,\beta)}. \tag{4.11}$$

Taking $L_x^{-1}T_t^{-1}$ to Eq. (4.11), which yields the solution $\phi(x,t)$ of Eq. (4.10)

$$\phi(x,t) = L_x^{-1}T_t^{-1}\left[\frac{p(\beta)\overline{g}(\alpha,\beta)}{p(\beta)-\lambda\overline{\psi}(\alpha,\beta)}\right]. \tag{4.12}$$

We illustrate the above method by the following example.

Example 4.1. *Solve*

$$\phi(x,t) = a - \lambda \int_0^x \int_0^t \phi(\delta,\epsilon) d\delta d\epsilon, \tag{4.13}$$

where a and λ are constants.

Applying the Laplace-General transform (LGT) to both sides of Eq. (4.13) and simplifying, we get

$$\overline{\phi}(\alpha,\beta) = a\frac{p(\beta)}{\alpha q(\beta) + \lambda}. \tag{4.14}$$

Therefore, we apply the inverse Laplace-General transform (LGT) to Eq. (4.14), which yields the solution

$$\phi(x,t) = aJ_0(2\sqrt{\lambda x t}).$$

4.2 Voltera Integro Partial-Differential Equations

Consider the Volterra integro-partial differential equation in the form

$$\frac{\partial \phi(x,t)}{\partial x} + \frac{\partial \phi(x,t)}{\partial t} = g(x,t) + \lambda \int_0^x \int_0^t \phi(x-\delta, t-\epsilon)\psi(\delta,\epsilon) d\delta d\epsilon, \tag{4.15}$$

with the conditions

$$\phi(x,0) = f_0(x), \phi(0,t) = h_0(t). \tag{4.16}$$

Applying LGT to both sides of Eq. (4.15) and single Laplace and single general transforms to Eq. (4.16) and simplifying, we have

$$\overline{\phi}(\alpha,\beta) = p(\beta)\left(\frac{\overline{g}(\alpha,\beta) + \overline{h}_0(\beta) + p(\beta)\overline{f}_0(\alpha)}{p(\beta)\alpha + p(\beta)q(\beta) - \lambda\overline{\psi}(\alpha,\beta)}\right). \tag{4.17}$$

Thus, we utilize the inverse LGT on Eq. (4.17), resulting in the solution

$$\phi(x,t) = L_x^{-1}T_t^{-1}\left[p(\beta)\left(\frac{\overline{g}(\alpha,\beta) + \overline{h}_0(\beta) + p(\beta)\overline{f}_0(\alpha)}{p(\beta)\alpha + p(\beta)q(\beta) - \lambda\overline{\psi}(\alpha,\beta)}\right)\right]. \tag{4.18}$$

We provide an example of the above procedure below.

Example 4.2. *Solve*

$$\frac{\partial\phi(x,t)}{\partial x}+\frac{\partial\phi(x,t)}{\partial t}=-1+e^x+e^t+e^{x+t}+\int_0^x\int_0^t\phi(x-\delta,t-\epsilon)d\delta d\epsilon \quad (4.19)$$

with conditions

$$\phi(x,0)=e^x=f_0(x),\ \phi(0,t)=e^t=h_0(t).$$

Substituting

$$\begin{array}{rcl}\overline{\psi}(\alpha,\beta) &=& =\dfrac{p(\beta)}{\alpha q(\beta)},\\ \overline{g}(\alpha,\beta) &=& -\dfrac{p(\beta)}{\alpha q(\beta)}+\dfrac{p(\beta)}{(\alpha-1)q(\beta)}+\dfrac{p(\beta)}{\alpha(q(\beta)-1)}+\dfrac{p(\beta)}{(\alpha-1)(q(\beta)-1)},\\ \overline{f_0}(\alpha) &=& \dfrac{1}{\alpha-1},\\ \overline{h_0}(\beta) &=& \dfrac{p(\beta)}{q(\beta)-1}\end{array},$$

in Eq. (4.18) and simplifying, we obtain the solution

$$\phi(x,t)=L_x^{-1}T_t^{-1}\left[\frac{p(\beta)}{(\alpha-1)(q(\beta)-1)}\right]=e^{x+t}.$$

4.3 Partial Integro -Differential Equations

In this section, we apply the Laplace – General transform (LGT) method to Partial Integro-Differential Equations.

$$\frac{\partial^2\phi(x,t)}{\partial t^2}-\frac{\partial^2\phi(x,t)}{\partial x^2}+\phi(x,t)+\int_0^x\int_0^t\psi(x-\delta,t-\epsilon)\phi(\delta,\epsilon)d\delta d\epsilon=g(x,t), \quad (4.20)$$

with conditions

$$\phi(x,0)=f_0(x),\frac{\partial\phi(x,0)}{\partial t}=f_1(x),\phi(0,t)=h_0(t),\frac{\partial\phi(0,t)}{\partial x}=h_1(t). \quad (4.21)$$

Taking LGT to both sides of Eq. (4.20) and single Laplace and single general transforms to Eq. (4.21) and simplifying, we have

$$\overline{\phi}(\alpha,\beta)=p(\beta)\left(\frac{\overline{g}(\alpha,\beta)+p(\beta)q(\beta)\overline{f}_0(\alpha)+p(\beta)\overline{f}_1(\alpha)-\alpha\overline{h}_0(\beta)-\overline{h}_1(\beta)}{p(\beta)q^2(\beta)-p(\beta)\alpha^2+p(\beta)+\overline{\psi}(\alpha,\beta)}\right). \quad (4.22)$$

Apply the inverse LGT to Eq. (4.22) yields the solution

$$\phi(x,t) = L_x^{-1}T_t^{-1}\left[p(\beta)\left(\frac{\overline{g}(\alpha,\beta)+p(\beta)q(\beta)\overline{f}_0(\alpha)+p(\beta)\overline{f}_1(\alpha)-\alpha\overline{h}_0(\beta)-\overline{h}_1(\beta)}{p(\beta)q^2(\beta)-p(\beta)\alpha^2+p(\beta)+\overline{\psi}(\alpha,\beta)}\right)\right]. \tag{4.23}$$

We illustrate the above method by an example.

Example 4.3. *Solve*

$$\frac{\partial^2\phi(x,t)}{\partial t^2}-\frac{\partial^2\phi(x,t)}{\partial x^2}+\phi(x,t)+\int_0^x\int_0^t e^{x-\delta+t-\epsilon}\phi(\delta,\epsilon)d\delta d\epsilon = e^{x+t}+xte^{x+t} \tag{4.24}$$

with conditions

$$\phi(x,0)=e^x,\ \frac{\partial\phi(x,0)}{\partial t}=e^x,\ \phi(0,t)=e^t,\ \frac{\partial\phi(0,t)}{\partial x}=e^t$$

Substituting

$$\begin{aligned}
\overline{\psi}(\alpha,\beta) &= \frac{p(\beta)}{(\alpha-1)(q(\beta)-1)},\\
\overline{g}(\alpha,\beta) &= \frac{p(\beta)}{(\alpha-1)(q(\beta)-1)}+\frac{p(\beta)}{(\alpha-1)^2(q(\beta)-1)^2},\\
\overline{f_0}(\alpha) &= \frac{1}{\alpha-1},\\
\overline{f_1}(\alpha) &= \frac{1}{\alpha-1},\\
\overline{h_0}(\beta) &= \frac{p(\beta)}{q(\beta)-1},\\
\overline{h_1}(\beta) &= \frac{p(\beta)}{q(\beta)-1},
\end{aligned}$$

in Eq. (4.18) and simplifying, we obtain the solution

$$\phi(x,t)=L_x^{-1}T_t^{-1}\left[\frac{p(\beta)}{(\alpha-1)(q(\beta)-1)}\right]=e^{x+t}.$$

5 Conclusion

The solution of a linear partial integro-differential equation was derived using the Laplace-General transform (LGT) approach. It is evident that this solution is consistent with the previous results obtained through the double Laplace transform and the Laplace-Sumudu transformation.

Acknowledgment. The authors are grateful to the reviewers for their valuable suggestions which improved the quality of the paper. This research is supported by Faculty of Science, Burapha University, grant no. SC04/2565.

References

[1] T. M. Elzaki, Double Laplace variational iteration method for solution of nonlinear convolution partial differential equations, Archives Des Sciences (ISI Journal), **65**, (2012), 588–593.

[2] T. M. Elzaki, A. A. Alderremy, On the new double integral transform for solving singular system of hyperbolic equations, Journal of Nonlinear Sciences and Applications, **11**, (2018), 1207–1214.

[3] S. A. Ahmed, T. M. Elzaki, A. A.Hassan, Solution of integral differential equations by new double integral transform (Laplace-Sumudu transform), Abstract and Applied Analysis, 1–7. https://doi.org/10.1155/2020/4725150

[4] H. Jafari, A new general integral transform for solving integral equations, Journal of Advanced Research, **32**, (2021), 133–138.

[5] L. Debnath, The double Laplace transforms and their properties with applications to functional, integral and partial differential equations. International Journal of Applied and Computational Mathematics, **2**, (2016), 223–242. https://doi.org/10.1007/s40819-015-0057-3