International Journal of Mathematics and Computer Science
Volume **20,** Issue no. 1, (2025), 447–458
DOI: https://doi.org/10.69793/ijmcs/01.2025/saif

Convex Normed Space

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(Received November 18,2024, Accepted December 18, 2024,
Published December 19, 2024)

Abstract

In this paper, the notion of convex absolute value is introduced as a generalization of absolute value and then used to introduce the notion of convex normed space as a generalization of normed space.

1 Introduction

In 2012, Kider [1] introduced a new fuzzy normed space; then, in 2019, Kider and Gheeab [2] introduced the general fuzzy normed space. In 2021, Khudhair and Kider [3] introduced the a-fuzzy normed space; then, in 2022, Khalafa and Kider [4] studied the linear operator of various types on a-fuzzy normed spaces. In 2024, Daher and Kider [5] introduced the convex fuzzy normed space, Kider [6] introduced the convex fuzzy metric space and Eidi, Hameed and Kider [7] introduced the convex fuzzy distance between two convex fuzzy compact set.
Here we introduce a generalization of normed space.

2 The convex absolute value

Definition 2.1. *Suppose that the function* $\mathbb{A}_{\mathbb{R}} : \mathbb{R} \to [0, \infty)$ *satisfies*

Key words and phrases: Convex normed space, Convex metric space, Convex bounded set, convex complete space, Convex continuous operator.
AMS (MOS) Subject Classifications: 46S99.
ISSN 1814-0432, 2025, https://future-in-tech.net

(i) $\mathbb{A}_{\mathbb{R}}(\lambda) \in [0, \infty)$; $\mathbb{A}_{\mathbb{R}}(\lambda) = 0 \iff \lambda = 0$,

(ii) $\mathbb{A}_{\mathbb{R}}(\lambda) \cdot \mathbb{A}_{\mathbb{R}}(\delta) = \mathbb{A}_{\mathbb{R}}(\lambda\delta)$,

(iii) $\mathbb{A}_{\mathbb{R}}(\lambda + \delta) \leq \sigma \mathbb{A}_{\mathbb{R}}(\lambda) + \mu \mathbb{A}_{\mathbb{R}}(\delta)$,

for all $0 < \sigma, \mu < 1$ *with* $\sigma + \mu = 1$ *and* $\forall \lambda, \delta \in \mathbb{R}$. *then we say that* $(\mathbb{R}, \mathbb{A}_{\mathbb{R}})$ *is a convex absolute value space (or c-AVS).*

Example 2.2. *For all* $\gamma \in \mathbb{R}$, *let* $\mathbb{A}_{\mathbb{R}}^{|\cdot|} : \mathbb{R} \to [0, \infty)$ *be defined by* $\mathbb{A}_{\mathbb{R}}^{|\cdot|}(\gamma) = \frac{1}{|\gamma|}$ *when* $\gamma \neq 0$ *and* $\mathbb{A}_{\mathbb{R}}^{|\cdot|}(0) = 0$. *Then* $(\mathbb{R}, \mathbb{A}_{\mathbb{R}}^{|\cdot|})$ *is c-AVS.*

To see this,

(i) $0 \leq \mathbb{A}_{\mathbb{R}}^{|\cdot|}(\gamma) < \infty$, and $\mathbb{A}_{\mathbb{R}}^{|\cdot|}(\gamma) = 0 \iff \gamma = 0$.

(ii) $\mathbb{A}_{\mathbb{R}}^{|\cdot|}(\gamma) \cdot \mathbb{A}_{\mathbb{R}}^{|\cdot|}(\lambda) = \frac{1}{|\gamma|} \cdot \frac{1}{|\lambda|} = \frac{1}{|\gamma \cdot \lambda|} = \mathbb{A}_{\mathbb{R}}^{|\cdot|}(\gamma \cdot \lambda)$.

(iii) $\sigma \mathbb{A}_{\mathbb{R}}^{|\cdot|}(\gamma) + \mu \mathbb{A}_{\mathbb{R}}^{|\cdot|}(\lambda) = \frac{\sigma}{|\gamma|} + \frac{\mu}{|\lambda|} = \frac{\sigma|\lambda| + \mu|\gamma|}{|\gamma||\lambda|} \geq \frac{1}{|\gamma + \lambda|} = \mathbb{A}_{\mathbb{R}}^{|\cdot|}(\gamma + \lambda)$.

for all $\sigma, \mu \in (0, 1)$ with $\sigma + \mu = 1$ and $\gamma, \lambda \in \mathbb{R}$.

Theorem 2.3. *Every c-AVS is an AVS.*

Proof.

If $(\mathbb{R}, \mathbb{A}_{\mathbb{R}})$ is a c-AVS, define $|\lambda| = \mathbb{A}_{\mathbb{R}}(\lambda)$ for each $\lambda \in \mathbb{R}$. Then $(\mathbb{R}, |\cdot|)$ is an AVS. Conditions (i) and (ii) follow directly. For (iii), since $|\gamma + \lambda| = \mathbb{A}_{\mathbb{R}}(\gamma + \lambda) \leq \sigma \mathbb{A}_{\mathbb{R}}(\gamma) + \mu \mathbb{A}_{\mathbb{R}}(\lambda) = \sigma|\gamma| + \mu|\lambda| \leq |\gamma| + |\lambda|$ for all $\sigma, \mu \in (0, 1)$ with $\sigma + \mu = 1$ and for all $\gamma, \lambda \in \mathbb{R}$, $(\mathbb{R}, |\cdot|)$ is an absolute value space.

Remark 2.4. *The converse of Theorem 2.3 is not true in general. For example, when* $(\mathbb{R}, |\cdot|)$ *is an AVS, by using* $\sigma|\gamma| + \mu|\lambda| \leq |\gamma + \lambda| \leq |\gamma| + |\lambda|$ *we see that condition (iii) is not satisfied for all* $\sigma, \mu \in (0, 1)$ *with* $\sigma + \mu = 1$ *and for all* $\gamma, \lambda \in \mathbb{R}$. *Hence,* $(\mathbb{R}, |\cdot|)$ *is not c-AVS.*

Definition 2.5. *Let* $(\mathbb{R}, \mathbb{A}_{\mathbb{R}})$ *be c-AVS and let* $\{\lambda_k\}_{k=1}^{\infty}$ *be a sequence in* $\mathbb{R}$. *Then* $\{\lambda_k\}_{k=1}^{\infty}$ *is approaches* $\lambda \in \mathbb{R}$ *as* $k \to \infty$ *if* $\forall \sigma > 0$, *we can find* $N \in \mathbb{N}$ *satisfying* $\mathbb{A}_{\mathbb{R}}(\lambda_k - \lambda) < \sigma$, *for all* $k \geq N$. *We write* $\lim_{(k \to \infty)} \lambda_k = \lambda$ *or* $\lambda_k \to \lambda$ *or* $\lim_{(k \to \infty)} \mathbb{A}_{\mathbb{R}}(\lambda_k - \lambda) = 0$.

Theorem 2.6. *Assume that* $(\mathbb{R}, \mathbb{A}_{\mathbb{R}})$ *is c-AVS and let* $\{\lambda_k\}_{k=1}^{\infty}$ *be a sequence in* $\mathbb{R}$. *If* $\lambda_k \to \lambda$ *and* $\lambda_k \to \gamma$, *then* $\lambda = \gamma$.

Proof.
Since $\lambda_k \to \lambda$ and $\lambda_k \to \gamma$, $\lim_{k\to\infty} \mathbb{A}_{\mathbb{R}}(\lambda_k - \lambda) = 0$ and $\lim_{k\to\infty} \mathbb{A}_{\mathbb{R}}(\lambda_k - \gamma) = 0$. Thus, $\mathbb{A}_{\mathbb{R}}[\lambda - \gamma] = \mathbb{A}_{\mathbb{R}}[\lambda - \lambda_k + \lambda_k - \gamma] < \sigma\mathbb{A}_{\mathbb{R}}(\lambda - \lambda_k) + \mu\mathbb{A}_{\mathbb{R}}(\lambda_k - \gamma)$ for all $\sigma, \mu \in (0, 1)$ with $\sigma + \mu = 1$. Therefore, $\lim_{k\to\infty} \mathbb{A}_{\mathbb{R}}(\lambda - \gamma) \leq \sigma \lim_{k\to\infty} \mathbb{A}_{\mathbb{R}}(\lambda_k - \lambda) + \mu \lim_{k\to\infty} \mathbb{A}_{\mathbb{R}}(\lambda_k - \gamma) = 0 + 0 = 0$. Hence, $\mathbb{A}_{\mathbb{R}}(\lambda - \gamma) = 0$ which implies that $\lambda - \gamma = 0$.

Definition 2.7. *If $(\mathbb{R}, \mathbb{A}_{\mathbb{R}})$ is c-AVS, then define $\mathbb{A}_{\mathbb{R}}(\alpha) = \mathbb{A}_{\mathbb{R}}(-\alpha)$ for all $\alpha \in \mathbb{R}$ and $\mathbb{A}_{\mathbb{R}}(1) = 1$.*

Definition 2.8. *Let $(\mathbb{R}, \mathbb{A}_{\mathbb{R}})$ be c-AVS and let $\{\lambda_k\}_{k=1}^{\infty} \in \mathbb{R}$. Then $\{\lambda_k\}_{k=1}^{\infty}$ is a Cauchy sequence if $\forall \sigma > 0$, we can find $N \in \mathbb{N}$ satisfying $\mathbb{A}_{\mathbb{R}}(\lambda_k - \lambda_j) < \sigma$, $\forall k, j \geq N$.*

Theorem 2.9. *Let $(\mathbb{R}, \mathbb{A}_{\mathbb{R}})$ be c-AVS and let $\{\lambda_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{R}$. If $\lambda_k \to \lambda$, then it is Cauchy.*

Proof.
Since $\lambda_k \to \lambda$ so $\forall \delta > 0$, $\exists N \in \mathbb{N}$ satisfying $\mathbb{A}_{\mathbb{R}}(\lambda_k - \lambda) < \delta$, $\forall k \geq N$. Thus, $\forall n, m \geq N$ $\mathbb{A}_{\mathbb{R}}(\lambda_k - \lambda_m) = \mathbb{A}_{\mathbb{R}}(\lambda_k - \lambda + \lambda - \lambda_m) \leq \mu\mathbb{A}_{\mathbb{R}}(\lambda_k - \lambda) + \sigma\mathbb{A}_{\mathbb{R}}(\lambda - \lambda_m)$ for all $\mu, \sigma \in (0, 1)$ with $\sigma + \mu = 1$. Therefore, $\mathbb{A}_{\mathbb{R}}(\lambda_k - \lambda_m) \leq \mu\delta + \sigma\delta = (\mu + \sigma)\delta = \delta$. Consequently, $\{\lambda_k\}_{k=1}^{\infty}$ is Cauchy.

Theorem 2.10. *Let $(\mathbb{R}, \mathbb{A}_{\mathbb{R}})$ be c-AVS and let $\{\lambda_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{R}$. If $\lambda_k \to \lambda$, then all $(\lambda_{k_n}) \subseteq \{\lambda_k\}_{k=1}^{\infty}$ satisfy $\lambda_{k_n} \to \lambda$.*

Proof.
$\lambda_k \to \lambda$ implies $\lim_{k\to\infty} \mathbb{A}_{\mathbb{R}}(\lambda_k - \lambda) = 0$. Also, $\{\lambda_k\}_{k=1}^{\infty}$ is a Cauchy sequence, so $\mathbb{A}_{\mathbb{R}}(\lambda_n - \lambda_m) \to 0$, when $n \to \infty$ and $m \to \infty$. Thus, $\mathbb{A}_{\mathbb{R}}[\lambda_{k_n} - \lambda] = \mathbb{A}_{\mathbb{R}}[\lambda_{k_n} - \lambda_k + \lambda_k - \lambda] \leq \sigma\mathbb{A}_{\mathbb{R}}[\lambda_{k_n} - \lambda_k] + \mu\mathbb{A}_{\mathbb{R}}[\lambda_k - \lambda]$ for all $\mu, \sigma \in (0, 1)$ with $\sigma + \mu = 1$. Therefore, $\lim_{k\to\infty} \mathbb{A}_{\mathbb{R}}[\lambda_{k_n} - \lambda] \leq \sigma \cdot 0 + \mu \cdot 0 = 0$. Consequently, $\lambda_{k_n} \to \lambda$.

3 The Convex Normed Space

Definition 3.1. *Let $\mathbb{V}$ be an $\mathbb{R}$-space over R, $(\mathbb{R}, \mathbb{A}_{\mathbb{R}})$ is c-AVS and let $\mathcal{N} : \mathbb{V} \to [0, \infty)$ be a function. If $\mathcal{N}$ satisfies*

(i) $0 \leq \mathcal{N}(y) < \infty$,

(ii) $\mathcal{N}(y) = 0$ *if and only if* $y = 0$,

(iii) $\mathcal{N}(\lambda y) = \mathbb{A}_{\mathbb{R}}(\lambda)\mathcal{N}(y)$, $\forall \lambda \in \mathbb{R}$. $\lambda \neq 0$,

(iv) $\mathcal{N}(y+g) \leq \gamma\mathcal{N}(y) + \delta\mathcal{N}(g)$. *where* $\gamma, \delta \in (0,1)$ *with* $\gamma + \delta = 1$, *for all* $y, g \in \mathbb{V}$,

then $(\mathbb{V}, \mathcal{N})$ *is a convex normed space (or c-NS).*

Example 3.2. *Define* $\mathcal{N}^{\|\cdot\|} : \mathbb{V} \to [0,\infty)$ *by:* $\mathcal{N}^{\|\cdot\|}(y) = \frac{1}{\|y\|}$ *if* $y \neq 0$ *and* $\mathcal{N}^{\|\cdot\|}(0) = 0$, $\forall y \in \mathbb{V}$. *Then* $(\mathbb{V}, \mathcal{N}^{\|\cdot\|})$ *is a c-NS when* $(\mathbb{V}, \|\cdot\|)$ *is a normed space. This space is called the c-NS induced by* $\|\cdot\|$.

Proof.

(i) $\mathcal{N}^{\|\cdot\|}(y) \in [0,\infty)$.

(ii) $\mathcal{N}^{\|\cdot\|}(y) = 0 \iff y = 0$.

(iii) $\mathbb{A}_{\mathbb{R}}(\alpha) \cdot \mathcal{N}^{\|\cdot\|}(y) = \frac{1}{|\alpha|} \cdot \frac{1}{\|y\|} = \frac{1}{\|\alpha y\|} = \mathcal{N}^{\|\cdot\|}(\alpha y)$.

(iv) $\gamma\mathcal{N}^{\|\cdot\|}(y) + \delta\mathcal{N}^{\|\cdot\|}(g) = \frac{\gamma}{\|y\|} + \frac{\delta}{\|g\|} = \frac{\gamma\|g\|+\delta\|y\|}{\|y\|\|g\|} \geq \frac{1}{\|y+g\|} = \mathcal{N}^{\|\cdot\|}(y+g)$.

Hence, $(\mathbb{V}, \mathcal{N}^{\|\cdot\|})$ is a c-NS, for any $\gamma, \delta \in (0,1)$ with $\gamma + \delta = 1$.

Remark 3.3.

(i) *If* $t, s \in [0,\infty)$, *then* $(\alpha t + (1-\alpha)s) \in [0,\infty)$ *for any* $\alpha \in (0,1)$, *or* $(\gamma t + \delta s) \in [0,\infty)$ *for any* $\gamma, \delta \in (0,1)$ *with* $\gamma + \delta = 1$. *In general, if* $t_1, t_2, \ldots, t_k \in [0,\infty)$, *then* $(\alpha_1 t_1 + \alpha_2 t_2 + \cdots + \alpha_k t_k) \in [0,\infty)$ *for any* $\alpha_1, \alpha_2, \ldots, \alpha_k \in (0,1)$ *with* $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$.

(ii) $\mathcal{N}(w_1 + w_2 + \cdots + w_k) \leq \alpha_1\mathcal{N}(w_1) + \alpha_2\mathcal{N}(w_2) + \cdots + \alpha_k\mathcal{N}(w_k)$, *for all* $w_1, w_2, \ldots, w_k \in U$ *and* $\alpha_1, \alpha_2, \ldots, \alpha_k \in (0,1)$ *with* $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$.

Proof.
The proof is elementary and so it is omitted.

Theorem 3.4. *If* $\mathcal{N} : \mathbb{R} \to [0,\infty)$ *defined by;* $\mathcal{N}(t) = \mathbb{A}_{\mathbb{R}}(t), \forall t \in \mathbb{R}$, *then* $(\mathbb{R}, \mathcal{N})$ *is c-NS.*

Proof.

(i) $\mathcal{N}(t) \in [0,\infty), \forall t \in \mathbb{R}$,

(ii) $\mathcal{N}(t) = 0 \iff$ if $\mathbb{A}_{\mathbb{R}}(t) = 0 \iff t = 0$,

(iii) $\mathcal{N}(t,s) = \mathbb{A}_{\mathbb{R}}(t,s) = \mathbb{A}_{\mathbb{R}}(t) \cdot \mathbb{A}_{\mathbb{R}}(s) = \mathcal{N}(t) \cdot \mathcal{N}(s), \forall t, s \in \mathbb{R}$ with $t \neq 0, s \neq 0$,

(iv) $\mathcal{N}(t+s) = \mathbb{A}_{\mathbb{R}}(t+s) \leq \gamma \mathbb{A}_{\mathbb{R}}(t) + \delta \mathbb{A}_{\mathbb{R}}(s) = \gamma \mathcal{N}(t) + \delta \mathcal{N}(s)$,

for any $\gamma, \delta \in (0,1)$ with $\gamma + \delta = 1$. Hence, $(\mathbb{R}, \mathcal{N})$ is c-NS.

Example 3.5. *If* $\mathbb{V} = C[d,b]$, *define* $\mathcal{N}(l) = \max_{\alpha \in [d,b]} \mathbb{A}_{\mathbb{R}}[l(\alpha)], \forall l \in \mathbb{V}$. *Then* $(\mathbb{V}, \mathcal{N})$ *is c-NS.*

Proof.

(i) $\mathcal{N}(l) \in [0, \infty), \forall l \in \mathbb{V}$.

(ii) $\mathcal{N}(l) = 0 \iff \max_{\alpha \in [a,b]} \mathbb{A}_{\mathbb{R}}[l(\alpha)] = 0 \iff \mathbb{A}_{\mathbb{R}}[l(\alpha)] = 0, \forall \alpha \in [d,b] \iff l(\alpha) = 0, \forall \alpha \in [d,b] \iff l = 0$.

(iii) $\mathcal{N}(tl) = \max_{\alpha \in [d,b]} \mathbb{A}_{\mathbb{R}}[tl(\alpha)] = \mathbb{A}_{\mathbb{R}}(t) \cdot \max_{\alpha \in [d,b]} \mathbb{A}_{\mathbb{R}}[l(\alpha)] = \mathbb{A}_{\mathbb{R}}(t) \cdot \mathcal{N}(l), \forall t \in \mathbb{R}$ with $t \neq 0$.

(iv) $\mathcal{N}(l+y) = \max_{\alpha \in [d,b]} \mathbb{A}_{\mathbb{R}}[(l+y)(\alpha)] = \max_{\alpha \in [d,b]} \mathbb{A}_{\mathbb{R}}[(l(\alpha) + y(\alpha))] \leq \max_{\alpha \in [d,b]} \gamma \mathbb{A}_{\mathbb{R}}[l(\alpha)] + \max_{\alpha \in [d,b]} \delta \mathbb{A}_{\mathbb{R}}[y(\alpha)]$ for any $\gamma, \delta \in (0,1)$ with $\gamma + \delta = 1$. Thus, $\mathcal{N}(l+y) \leq \gamma \max_{\alpha \in [a,b]} \mathbb{A}_{\mathbb{R}}[l(\alpha)] + \delta \max_{\alpha \in [a,b]} \mathbb{A}_{\mathbb{R}}[y(\alpha)] = \gamma \mathcal{N}(l) + \delta \mathcal{N}(y)$.

Therefore, $(\mathbb{V}, \mathcal{N})$ is c-NS.

Theorem 3.6. *Every c-NS is a NS.*

Proof.
If $(\mathbb{V}, \mathcal{N})$ is a c-NS, define $\|y\| = \mathcal{N}(y)$ for all $y \in \mathbb{V}$. Then conditions (i), (ii), and (iii) are simple.

(iv) $\|y+g\| = \mathcal{N}(y+g) \leq \gamma \mathcal{N}(y) + \delta \mathcal{N}(g) = \gamma \|y\| + \delta \|g\| \leq \|y\| + \|g\|$ for all $\gamma, \delta \in (0,1)$ with $\gamma + \delta = 1$ and for all $u, v \in \mathbb{V}$. Thus, $\|y+g\| \leq \|y\| + \|g\|$. Hence, $(\mathbb{V}, \|\cdot\|)$ is a normed space.

Remark 3.7. *The converse of Theorem 3.6 is not true in general. For example, if* $(\mathbb{V}, \|\cdot\|)$ *is a normed space, then since* $\gamma \|y\| + \delta \|g\| \leq \|y+g\| \leq \|y\| + \|g\|$, *condition (iv) is not satisfied, since* $\|y+g\| \geq \gamma \|y\| + \delta \|g\|$, *for all* $\gamma, \delta \in (0,1)$ *with* $\gamma + \delta = 1$ *and for all* $y, g \in \mathbb{V}$. *Hence,* $(\mathbb{V}, \|\cdot\|)$ *is not c-NS.*

Theorem 3.8. *If* $(\mathbb{V}_1, \mathcal{N}_1)$ *and* $(\mathbb{V}_2, \mathcal{N}_2)$ *are two c-NS, then* $(\mathbb{V}, \mathcal{N})$ *is c-NS where* $\mathbb{V} = \mathbb{V}_1 \times \mathbb{V}_2$ *and* $\mathcal{N}[(y_1, y_2)] = \gamma \mathcal{N}_1(y_1) + \delta \mathcal{N}_2(y_2)$ *for all* $(y_1, y_2) \in \mathbb{V}$, *for all* $\gamma, \delta \in (0,1)$ *with* $\gamma + \delta = 1$.

Proof.

(i) Since $0 \leq \mathcal{N}_1(y_1) < \infty$ and $0 \leq \mathcal{N}_2(y_2) < \infty$, $0 \leq \mathcal{N}[(y_1, y_2)] < \infty$

(ii) $\mathcal{N}[(y_1, y_2)] = 0$ if and only if $\gamma\mathcal{N}_1(y_1) + \delta\mathcal{N}_2(y_2) = 0$ if and only if $\mathcal{N}_1(y_1) = 0$ and $\mathcal{N}_2(y_2) = 0$ if and only if $y_1 = 0$ and $y_2 = 0 \iff (y_1, y_2) = (0, 0)$.

(iii) $\mathcal{N}[\alpha(y_1, y_2)] = \mathcal{N}[(\alpha y_1, \alpha y_2)] = \gamma\mathcal{N}_1(\alpha y_1)+\delta\mathcal{N}_2(\alpha y_2) \leq \mathbb{A}_{\mathbb{R}}(\alpha)\cdot\gamma\mathcal{N}_1(\alpha y_1)+ \mathbb{A}_{\mathbb{R}}(\alpha) \cdot \delta\mathcal{N}_2(y_2) \leq \mathbb{A}_{\mathbb{R}}(\alpha)[\gamma\mathcal{N}_1(y_1) + \delta\mathcal{N}_2(y_2)] = \mathbb{A}_{\mathbb{R}}(\alpha) \cdot \mathcal{N}[(y_1, y_2)]$.

(iv) $\mathcal{N}[(y_1, y_2) + (g_1, g_2)] = \mathcal{N}[(y_1 + g_1) + (y_2 + g_2)] = \gamma\mathcal{N}_1(y_1 + g_1) + \delta\mathcal{N}_2(y_2 + g_2) \leq \gamma[\sigma\mathcal{N}_1(y_1) + \theta\mathcal{N}_1(g_1)] + \delta[\sigma\mathcal{N}_2(y_2) + \theta\mathcal{N}_2(g_2)]$ where $\gamma + \delta = 1$ for all $\gamma, \delta \in (0, 1)$ and $\sigma + \theta = 1$ for all $\sigma, \theta \in (0, 1)$. Thus, $\mathcal{N}[(y_1, y_2) + (g_1, g_2)] \leq \sigma[\gamma\mathcal{N}_1(y_1) + \delta\mathcal{N}_2(y_2)] + \theta[\gamma\mathcal{N}_1(g_1) + \delta\mathcal{N}_2(g_2)] = \sigma\mathcal{N}[(y_1, y_2)] + \theta\mathcal{N}[(g_1, g_2)]$.

Hence, $(\mathbb{V}, \mathcal{N})$ is c-NS.

The proof of the following results can be established by following a similar technique as thaT of the proof of Theorem 3.8.

Corollary 3.9. *If $(\mathbb{V}_1, \mathcal{N}_1), (\mathbb{V}_2, \mathcal{N}_2), \ldots, (\mathbb{V}_k, \mathcal{N}_k)$ are c-NS, then $(\mathbb{V}, \mathcal{N})$ is c-NS where $\mathbb{V} = \mathbb{V}_1 \times \mathbb{V}_2 \times \cdots \times \mathbb{V}_k$ and $\mathcal{N}[(y_1, y_2, \ldots, y_k)] = \delta_1\mathcal{N}_1(y_1) + \delta_2\mathcal{N}_2(y_2)+\cdots+\delta_k\mathcal{N}_k(y_k)$, for all $(y_1, y_2, \ldots, y_k) \in \mathbb{V}$, where $\delta_1+\delta_2+\cdots+\delta_k = 1$, for all $\delta_1, \delta_2, \ldots, \delta_k \in (0, 1)$.*

Corollary 3.10. *If $(\mathbb{V}, \mathcal{N})$ is c-NS, then $(\mathbb{V}^k, \mathcal{N}_{\mathbb{V}})$ is c-NS where $\mathbb{V}^k = \mathbb{V} \times \mathbb{V} \times \cdots \times \mathbb{V}$ (k-times) and $\mathcal{N}_{\mathbb{V}}[(y_1, y_2, \ldots, y_k)] = \delta_1\mathcal{N}(y_1) + \delta_2\mathcal{N}(y_2) + \cdots + \delta_k\mathcal{N}(y_k)$ for all $(y_1, y_2, \ldots, y_k) \in \mathbb{V}$, where $\delta_1 + \delta_2 + \cdots + \delta_k = 1$, for all $\delta_1, \delta_2, \ldots, \delta_k \in (0, 1)$.*

Definition 3.11. *If $(\mathbb{V}, \mathcal{N})$ is c-NS, then*

(i) For any $y \in \mathbb{V}$, let $c - \mathbb{B}(y, \alpha) = \{v \in \mathbb{V} : \mathcal{N}(y - v) < \alpha\}$. Then $c - \mathbb{B}(y, \alpha)$ is a convex open ball with the center $y \in \mathbb{V}$ and radius $\alpha > 0$.

(ii) $\mathbb{W} \subseteq \mathbb{V}$ is a convex open set (or simply c-OS) if $c - \mathbb{B}(w, \alpha) \subseteq \mathbb{W}$ for any $w \in \mathbb{W}$ and for some $\alpha > 0$.

Theorem 3.12. *If $c - \mathbb{B}(y, \alpha)$ is a convex open ball in c-NS $(\mathbb{V}, \mathcal{N})$, then it is a c-OS.*

Proof.
Let $c-\mathbb{B}(y,\alpha)$ be a convex open ball, where $y \in \mathbb{V}$ and $\alpha > 0$. If $v \in c-\mathbb{B}(y,\alpha)$, then $\mathcal{N}(y-v) < \alpha$. Put $\beta = \mathcal{N}(y-v)$. Then $\beta < \alpha$ and there exists $\sigma > 0$ such that $(\gamma\beta + \delta\sigma) < \alpha$, where $\gamma, \delta \in (0,1)$ with $\gamma + \delta = 1$. Consider $c-\mathbb{B}(v,\sigma)$. To prove $c-\mathbb{B}(v,\sigma) \subseteq c-\mathbb{B}(y,\alpha)$, let $z \in c-\mathbb{B}(v,\sigma)$. So, $\mathcal{N}(v-z) < \sigma$. Hence, $\mathcal{N}(y-z) \leq \gamma\mathcal{N}(y-v)+\delta\mathcal{N}(v-z)$, where $\delta \in (0,1)$ with $\gamma+\delta = 1$. Thus, $\mathcal{N}(y-z) \leq \gamma\beta + \delta\sigma$, so $z \in c-\mathbb{B}(y,\alpha)$. That is, $c-\mathbb{B}(v,\sigma) \subseteq c-\mathbb{B}(y,\alpha)$. Therefore, $c-\mathbb{B}(y,\alpha)$ is a c-OS.

Definition 3.13.

(i) A subset $\mathbb{D}$ of a c-NS $(\mathbb{V},\mathcal{N})$ is convex closed if $\mathbb{V} \setminus \mathbb{D}$ is a c-OS.

(ii) The convex closure $\bar{\mathbb{E}}$ of a subset of a c-NS $(\mathbb{V},\mathcal{N})$ is $\bar{\mathbb{E}} = \bigcap\{\mathbb{D} : \mathbb{D}$ is convex closed in $\mathbb{V}$ and $\mathbb{E} \subseteq \mathbb{D}\}$.

The proof of the following lemma is clear.

Lemma 3.14. *Let $(\mathbb{V},\mathcal{N})$ be a c-NS.*
(i) If $\mathbb{G}_1, \mathbb{G}_2, \ldots, \mathbb{G}_k, \ldots$ are c-OS in $\mathbb{V}$, then $\bigcup_{j=1}^{\infty} \mathbb{G}_j$ is c-OS in $\mathbb{V}$.
(ii) If $\mathbb{G}_1, \mathbb{G}_2, \ldots, \mathbb{G}_k$ are c-OS in $\mathbb{V}$, then $\bigcap_{j=1}^{k} \mathbb{G}_j$ is c-OS in $\mathbb{V}$.

Definition 3.15. *Let $(\mathbb{V},\mathcal{N})$ be c-NS and let $(y_k) \in \mathbb{V}$. Then (y_k) is convex converges to $y \in \mathbb{V}$ when $k \to \infty$ (or simply $u_k \to u$) if $\forall$, $c-\mathbb{B}(y,t)$, $\exists M$ such that $y_k \in c-\mathbb{B}(y,t)$, $\forall k \geq M$ where $t > 0$. This is equivalent to $\forall t > 0$, $\exists M \in \mathbb{N}$ satisfies $\mathcal{N}(y_k - y) < t$, $\forall k \geq M$ or $\lim_{k\to\infty} y_k = y$ or $\lim_{n\to\infty} \mathcal{N}(y_k - y) = 0$.*

Lemma 3.16. *In c-NS $(\mathbb{V},\mathcal{N})$ if $u_k \to p$ and $u_k \to q$, then $p = q$.*

Proof.
Since $u_k \to p$ and $u_k \to q$, $\mathcal{N}(u_k - p) \to 0$ and $\mathcal{N}(u_k - q) \to 0$ as $k \to \infty$. Thus, $\mathcal{N}(p-q) = \mathcal{N}(p-u_k+u_k-q) \leq \gamma\mathcal{N}(p-u_k) + \delta\mathcal{N}(u_k-q)$ where $\gamma, \delta \in (0,1)$ with $\gamma+\delta = 1$. Therefore, $\mathcal{N}(p-q) \leq \gamma \lim_{k\to\infty} \mathcal{N}(p-u_k) + \delta \lim_{k\to\infty} \mathcal{N}(u_k - q) = \gamma \cdot 0 + \delta \cdot 0 = 0$. As a result, $\mathcal{N}(p-q) = 0$ which implies that $p = q$.

Definition 3.17. *If $(\mathbb{V},\mathcal{N})$ is c-NS, then*

(i) $\mathbb{Y} \subset \mathbb{V}$ is convex bounded (or simply CB) if $\exists\, \lambda > 0$, satisfying $\mathcal{N}(p) < \lambda$, $\forall p \in \mathbb{Y}$. Otherwise, $\mathbb{Y}$ is not CB.

(ii) A sequence (u_k) in a c-NS $(U,\mathcal{N})$ is CB if $\exists\, \lambda > 0$, satisfying $\mathcal{N}(u_k) < \lambda$, $\forall k \in \mathbb{N}$. Otherwise, (u_k) is not CB.

Proposition 3.18. *In a c-NS* $(\mathbb{V}, \mathcal{N})$,

(i) *if* $\mathbb{G}$ *is a CB subset of* $\mathbb{V}$, *then* $\lambda\mathbb{G}$ *is CB for every* $\lambda \neq 0 \in \mathbb{R}$.

(ii) *if* (y_k) *is a CB sequence in* $\mathbb{V}$, *then* (αy_k) *is CB for every* $\alpha \neq 0 \in \mathbb{R}$.

(iii) *If* $\mathbb{G}$ *and* $\mathbb{W}$ *are CB subsets of* $\mathbb{V}$, *then* $\mathbb{G} + \mathbb{W}$ *is CB.*

(iv) *If* (g_k) *and* (y_k) *are two CB sequences in* $\mathbb{V}$, *then* $(g_k + y_k)$ *is CB.*

Proof.

(i) Since $\mathbb{G}$ is a CB subset of $\mathbb{V}$, there is $\mu > 0$ such that $\mathcal{N}(g) < \mu$ for all $g \in \mathbb{G}$. Thus, $\mathcal{N}(\lambda g) \leq \mathbb{A}_{\mathbb{R}}(\lambda)\mathcal{N}(g) < \mathbb{A}_{\mathbb{R}}(\lambda)\mu < \beta$ with $\beta > 0$. Therefore, $\lambda\mathbb{G}$ is CB.

(ii) If (y_k) is a CB sequence in $\mathbb{V}$, then there exists $\mu > 0$ such that $\mathcal{N}(y_k) < \mu$, $\forall k \in \mathbb{N}$. Thus, $\mathcal{N}(\alpha y_k) \leq \mathbb{A}_{\mathbb{R}}(\alpha)\mathcal{N}(y_k) < \mathbb{A}_{\mathbb{R}}(\alpha)\mu < \beta$ with $\beta > 0$. Therefore, (αy_k) is CB.

(iii) Since $\mathbb{G}$ and $\mathbb{W}$ are CB, $\exists\, \mu > 0$, $\epsilon > 0$ satisfying $\mathcal{N}(l) < \mu$ as well as $\mathcal{N}(f) < \epsilon$, $\forall l \in \mathbb{G}$, $f \in \mathbb{W}$. Thus, $\mathcal{N}(l+f) \leq \delta\mathcal{N}(l) + \gamma\mathcal{N}(f) < \delta\mu + \gamma\epsilon$ with $\delta+\gamma = 1$. So $\exists\, \lambda > 0$ satisfying $(\delta\mu+\gamma\epsilon) < \lambda$. Hence, $\mathcal{N}(l+f) < \lambda$, $\forall (l+f) \in (\mathbb{G} + \mathbb{W})$. Thus, $\mathbb{G} + \mathbb{W}$ is CB.

(iv) Since (g_k) and (y_k) are two CB sequences in $\mathbb{V}$, $\exists\, \mu > 0$, $\epsilon > 0$ satisfying $\mathcal{N}(g_k) < \mu$ as well as $\mathcal{N}(y_k) < \epsilon$, $\forall k \in \mathbb{N}$. Thus, $\mathcal{N}(g_k + y_k) \leq \delta\mathcal{N}(g_k) + \gamma\mathcal{N}(y_k) < \delta\mu + \gamma\epsilon$ with $\delta + \gamma = 1$. So $\exists\, \lambda > 0$ satisfying $(\delta\mu + \gamma\epsilon) < \lambda$. Hence, $\mathcal{N}(g_k + y_k) < \lambda$, $\forall k \in \mathbb{N}$.

Therefore, $(g_k + y_k)$ is CB.

Proposition 3.19. *Let* (y_k) *be a sequence in a c-NS* $(\mathbb{V}, \mathcal{N})$. *If* $y_k \to y \in \mathbb{V}$, *then* (y_k) *is CB.*

Proof.
Since $y_k \to y \in \mathbb{V}$, $\forall\, \alpha > 0$, $\exists N \in \mathbb{N}$ satisfying $\mathcal{N}(y_k - y) < \alpha$, for all $k \geq N$. Thus, $\mathcal{N}(y_k) \leq \delta_1\mathcal{N}(y_k - y) + \delta_2\mathcal{N}(y) < \delta_1\alpha + \delta_2\mathcal{N}(y)$,where $\delta_1 + \delta_2 = 1$. Then, put $\delta_1\alpha + \delta_2\mathcal{N}(y) = \beta$ for some $\beta > 0$, which follows that $\mathcal{N}(y_k) < \beta$ for all $k \in \mathbb{N}$. Hence, (y_k) is CB.

Definition 3.20. *Let* $(\mathbb{V}, \mathcal{N})$ *be c-NS,* $(y_k) \in \mathbb{V}$, (y_k) *is convex Cauchy in* $\mathbb{V}$ *if* $\forall\, \epsilon > 0$, $\exists N \in \mathbb{N}$ *satisfying* $\mathcal{N}(y_k - y_m) < \epsilon$, $\forall k, m \geq N$.

The proof of the following lemma is immediate.

Lemma 3.21. *Let $\mathbb{A}_{\mathbb{R}}[\mathcal{N}(y)] = \mathcal{N}(y)$ for all $y \in \mathbb{V}$. When $(\mathbb{V}, \mathcal{N})$ is c-NS, so $\mathbb{A}_{\mathbb{R}}[\mathcal{N}(y) - \mathcal{N}(g)] \leq \mathcal{N}(y - g)$ for all $y, g \in \mathbb{V}$.*

Lemma 3.22. *If $(\mathbb{A}, \mathcal{N})$ is c-FN, then $\mathcal{N}(y - g) = \mathcal{N}(g - y)$, $\forall y, g \in \mathbb{V}$.*

Proof.
$\mathcal{N}(y-g) = \mathcal{N}[(-1)(g-y)] \leq \mathbb{A}_{\mathbb{R}}(-1) \cdot \mathcal{N}(g-y) = \mathbb{A}_{\mathbb{R}}(1) \cdot \mathcal{N}(g-y) = \mathcal{N}(g-y)$ so $\mathcal{N}(u-v) \leq \mathcal{N}(v-u)$. By using a similar technique, $\mathcal{N}(v-u) \leq \mathcal{N}(u-v)$. Thus, $\mathcal{N}(y - g) = \mathcal{N}(g - y)$, $\forall y, g \in \mathbb{V}$.

Definition 3.23. *Let $\neq \emptyset$, if $\mathbb{M} : \mathbb{V} \times \mathbb{V} \to [0, \infty)$ satisfies*

(i) $\mathbb{M}(y, g) \in [0, \infty)$.

(ii) $y = g \iff \mathbb{M}(y, g) = 0$.

(iii) $\mathbb{M}(y, g) = \mathbb{M}(g, y)$.

(iv) $\gamma\mathbb{M}(y, b) + \delta\mathbb{M}(b, g) \geq \mathbb{M}(y, g)$, $\forall 0 < \gamma, \delta < 1$ *with* $\gamma + \delta = 1$ *and* $\forall y, g, w \in \mathbb{V}$.

Then $(\mathbb{V}, \mathbb{M})$ is a convex metric space (or simply c-MS).

The next theorem is easy to prove.

Theorem 3.24. *If $(\mathbb{V}, \mathcal{N})$ is c-NS, then $(\mathbb{V}, \mathbb{M}_{\mathcal{N}})$ is c-MS, where $\mathbb{M}_{\mathcal{N}}(y, g) = \mathcal{N}(y - g)$, for all $y, g \in \mathbb{V}$.*

Definition 3.25. *If $(\mathbb{V}, \mathcal{N})$ is c-NS, then $\mathbb{D} \subseteq \mathbb{V}$ is known as convex dense in $\mathbb{V}$ if whenever $\overline{\mathbb{D}} = \mathbb{V}$.*

Theorem 3.26. *If $(\mathbb{V}, \mathcal{N})$ is a c-NS, then it is a topological space.*

Proof.
If $(\mathbb{V}, \mathcal{N})$ is a c-NS, then put $\mathbb{T}_{\mathcal{N}} = \{\mathbb{W} \subseteq \mathbb{V} : \mathbb{W} \text{ is c-OS in } \mathbb{V}\}$.

(i) $\emptyset, \mathbb{V} \in \mathbb{T}_{\mathcal{N}}$.

(ii) If $\{\mathbb{E}_i : i \in I\} \in \mathbb{T}_{\mathcal{N}}$, then $\bigcup_{i \in I} \mathbb{E}_i \in \mathbb{T}_{\mathcal{N}}$ by Lemma 3.14.

(iii) Let $\mathbb{E}_1, \mathbb{E}_2, \ldots, \mathbb{E}_k \in \mathbb{T}_{\mathcal{N}}$, then $\bigcap_{i=1}^{k} \mathbb{E}_i \in \mathbb{T}_{\mathcal{N}}$ by Lemma 3.14.

Hence, $(\mathbb{V}, \mathbb{T}_{\mathcal{N}})$ is a topological space.

Definition 3.27. *In c-NS, $(\mathbb{V}, \mathcal{N})$ is convex complete if $\forall$ convex Cauchy sequence $(y_k) \in \mathbb{V}$, then $\exists y \in \mathbb{V}$, satisfying $y_k \to y$.*

Theorem 3.28. *In c-NS, $(\mathbb{V}, \mathcal{N})$, if $y_k \to y \in \mathbb{V}$, then (y_k) is convex Cauchy.*

Proof.
Let $(y_k) \in \mathbb{V}$ with $y_k \to y \in \mathbb{V}$. So, $\forall\, 0 < \sigma < 1$, $\exists N$ satisfying $\mathcal{N}(y_k - y) < \sigma$, $\forall k \geq N$. Thus, for each $m, k \geq N$, $\mathcal{N}(y_k - y_m) \leq \gamma \mathcal{N}(y_k - y) + \delta \mathcal{N}(y - y_m)$, where $\gamma, \delta \in (0, 1)$ with $\gamma + \delta = 1$. Therefore, $\mathcal{N}(y_k - y_m) < \alpha\sigma + \beta\sigma = \sigma$. Hence, (u_k) is a convex fuzzy Cauchy sequence.

Theorem 3.29. *In c-NS, $(\mathbb{V}, \mathcal{N})$ if $\mathbb{D} \subset \mathbb{V}$, then the following statements are equivalent*

(i) $d \in \overline{\mathbb{D}}$.

(ii) $\exists (d_k) \in \mathbb{D}$ *with* $d_k \to d$.

Proof.
(ii) $\Rightarrow$ (i): If $(d_k) \in \mathbb{D}$ with $d_k \to d$, then $d \in \mathbb{D}$ or for every $c - \mathbb{B}(d, \lambda)$, $d_k \neq d$, and so d is a limit of $\mathbb{D}$. Therefore, $d \in \overline{\mathbb{D}}$.
(i) $\Rightarrow$ (ii): Let $d \in \overline{\mathbb{D}}$. If $d \in \mathbb{D}$, then $(d, d, \ldots, d, \ldots)$ is the desired sequence. If $d \notin \mathbb{D}$, then $(d_k) \in \mathbb{D}$ since $\mathcal{N}(d_k - d) < \frac{1}{k}$ for each $k = 1, 2, 3, \ldots$. Thus, $c - \mathbb{B}(d, \frac{1}{k})$ contains $d_k \in \mathbb{D}$ and $d_k \to d$.

4 Convex continuous and uniform convex continuous operators

Definition 4.1. *If $(V, \mathcal{N}_{\mathbb{V}})$ and $(Y, \mathcal{N}_{\mathbb{Y}})$ are two c-NS, then*

(i) *$\mathcal{T} : \mathbb{V} \to \mathbb{Y}$ is convex continuous at $v \in \mathbb{V}$ if $\forall\, \alpha > 0$, $\exists\, \beta > 0$ such that $\mathcal{N}_{\mathbb{V}}(v - y) < \beta$ implies $\mathcal{N}_{\mathbb{Y}}[\mathcal{T}(v) - \mathcal{T}(y)] < \alpha$, $\forall\, y \in \mathbb{Y}$. If this is true $\forall\, v \in \mathbb{V}$, then $\mathcal{T}$ is convex continuous on $\mathbb{V}$.*

(ii) *$\mathcal{T} : \mathbb{V} \to \mathbb{Y}$ is strongly convex continuous at $v \in \mathbb{V}$ if $\mathcal{N}_{\mathbb{V}}(v - y) < \mathcal{N}_{\mathbb{Y}}[\mathcal{T}(v) - \mathcal{T}(y)]$, $\forall\, y \in \mathbb{Y}$. If this is true $\forall\, v \in \mathbb{V}$, then $\mathcal{T}$ is strongly convex continuous on $\mathbb{V}$.*

Proposition 4.2. *Every strongly convex continuous operator on* $\mathbb{V}$ *is convex continuous on* $\mathbb{V}$ *whenever* $(\mathbb{V}, \mathcal{N}_{\mathbb{V}})$ *and* $(\mathbb{Y}, N_{\mathbb{Y}})$ *are two c-NS.*

Proof.
If $\mathcal{T} : \mathbb{V} \to \mathbb{Y}$ is a strongly convex continuous operator on $\mathbb{V}$, then for any $v \in \mathbb{V}$ and any $y \in \mathbb{Y}$, $\mathcal{N}_{\mathbb{V}}(v - y) < \mathcal{N}_{\mathbb{Y}}[\mathcal{T}(v) - \mathcal{T}(y)]$ for . Thus, for every $\alpha > 0$ with $\mathcal{N}_{\mathbb{Y}}[\mathcal{T}(v) - \mathcal{T}(y)] < \alpha$, we can find some $\beta > 0$ satisfying $\mathcal{N}_{\mathbb{V}}(v - y) < \beta < \mathcal{N}_{\mathbb{Y}}[\mathcal{T}(v) - \mathcal{T}(y)]$ for any $y \in \mathbb{Y}$. Hence, $\mathcal{T} : \mathbb{V} \to \mathbb{Y}$ is a convex continuous operator on v since v was an arbitrary point of $\mathbb{V}$.

The proof of the following theorem is easy.

Theorem 4.3. *If* $(\mathbb{V}, \mathcal{N}_{\mathbb{V}})$ *and* $(\mathbb{Y}, \mathcal{N}_{\mathbb{Y}})$ *are two c-NS, then the operator* $\mathcal{T} : \mathbb{V} \to \mathbb{Y}$ *is convex continuous at* $y \in \mathbb{V}$ *if and only if* $\mathcal{T}(y_k) \to \mathcal{T}(y) \in Y$ *whenever* $y_k \to y \in \mathbb{V}$.

Theorem 4.4. *If* (y_k) *is a sequence in* $\mathbb{V}$ *with* $y_k \to y$*, then* $\mathcal{N}(y_k) \to \mathcal{N}(y)$ *when* $(\mathbb{V}, \mathcal{N})$ *is c-NS.*

Proof.
If $(y_k) \in \mathbb{V}$ with $y_k \to y$, then $\lim_{k\to\infty} \mathcal{N}(y_k - y) = 0$. Thus, $\mathbb{A}_{\mathbb{R}}[\mathcal{N}(y_k) - \mathcal{N}(y)] \leq \mathcal{N}(y_k - y)$. Hence, $\lim_{k\to\infty} \mathbb{A}_{\mathbb{R}}[\mathcal{N}(y_k) - \mathcal{N}(y)] \leq \lim_{k\to\infty} \mathcal{N}(y_k - y) = 0$. Consequently, $\mathcal{N}(y_k) \to \mathcal{N}(y)$.
The proof of the following theorem is easy.

Theorem 4.5. *If* $(\mathbb{V}, \mathcal{N}_{\mathbb{V}})$ *and* $(\mathbb{Y}, \mathcal{N}_{\mathbb{Y}})$ *are two c-NS, then the following are equivalent:*

(i) *The operator* $\mathcal{T} : \mathbb{V} \to \mathbb{Y}$ *is convex continuous at* $v \in \mathbb{V}$.

(ii) $\mathcal{T}^{-1}(P)$ *is c-OS in* $\mathbb{V}$ *for all c-OS subset* $\mathbb{P}$ *of* $\mathbb{Y}$.

(iii) $\mathcal{T}^{-1}(\mathbb{E})$ *is convex closed in* $\mathbb{V}$ *for all convex closed* $\mathbb{E} \subset \mathbb{Y}$.

Definition 4.6. *If* $(\mathbb{V}, \mathcal{N}_{\mathbb{V}})$ *and* $(\mathbb{Y}, \mathcal{N}_{\mathbb{Y}})$ *are c-NS, then*

(i) $\mathcal{T} : \mathbb{V} \to \mathbb{Y}$ *is uniformly convex continuous on* $\mathbb{V}$*, if* $\forall\, 0 < \alpha$, $\exists\, \beta$, $0 < \beta$ *with* $\mathcal{N}_{\mathbb{Y}}[\mathcal{T}(v) - \mathcal{T}(y)] < \alpha$ *whenever* $\mathcal{N}_{\mathbb{V}}(v - y) < \beta$*, for all* $v, y \in \mathbb{V}$.

(ii) $\mathcal{T} : \mathbb{V} \to \mathbb{Y}$ *is strong uniformly convex continuous on* $\mathbb{V}$*, if* $\mathcal{N}_{\mathbb{V}}(v - y) < \mathcal{N}_{\mathbb{Y}}[\mathcal{T}(v) - \mathcal{T}(y)]$*, for all* $v, y \in \mathbb{V}$.

Proposition 4.7. *If* $(\mathbb{V}, \mathcal{N}_{\mathbb{V}})$ *and* $(\mathbb{Y}, \mathcal{N}_{\mathbb{Y}})$ *are c-NS, then every strongly uniformly convex continuous operator on* $\mathbb{V}$ *is uniformly convex continuous on* $\mathbb{V}$.

Proof.
If $\mathcal{T}: \mathbb{V} \to \mathbb{Y}$ is strongly uniformly convex continuous on $\mathbb{V}$, then $\mathcal{N}_{\mathbb{V}}(v-y) < \mathcal{N}_{\mathbb{Y}}[\mathcal{T}(v) - \mathcal{T}(y)]$, for all $v, y \in \mathbb{V}$. Thus, for every $0 < \alpha$, with $\mathcal{N}_{\mathbb{Y}}[\mathcal{T}(v) - \mathcal{T}(y)] < \alpha$, we can find some $0 < \beta$ satisfying $\mathcal{N}_{\mathbb{V}}(v-y) < \beta < \mathcal{N}_{\mathbb{Y}}[\mathcal{T}(v) - \mathcal{T}(y)]$ for any $v, y \in \mathbb{V}$. Hence, $\mathcal{T}: \mathbb{V} \to \mathbb{Y}$ is a uniformly convex continuous operator on $\mathbb{V}$ since v and y were arbitrary points of $\mathbb{V}$.

Theorem 4.8. *Assume $(\mathbb{V}, \mathcal{N}_{\mathbb{V}})$ and $(\mathbb{Y}, \mathcal{N}_{\mathbb{Y}})$ are c-NS and the operator $\mathcal{T}: \mathbb{V} \to \mathbb{Y}$ is uniformly convex continuous on $\mathbb{V}$. Then $(\mathcal{T}(y_k))$ is a convex Cauchy sequence in $\mathbb{Y}$ if (y_k) is a convex Cauchy sequence in $\mathbb{V}$.*

Proof.
$\forall \alpha$, $0 < \alpha$, $\exists \beta$, $0 < \beta$ such that $\mathcal{N}_{\mathbb{V}}(g-y) < \beta$ implies $\mathcal{N}_{\mathbb{Y}}[\mathcal{T}(g) - \mathcal{T}(y)] < \alpha$, $\forall g, y \in \mathbb{V}$. But (y_k) is convex Cauchy. Thus, $\forall 0 < \beta$, $\exists N \in \mathbb{N}$ such that $\mathcal{N}_{\mathbb{V}}(y_k - y_m) < \beta$, $\forall m, k \geq N$. Hence, $\mathcal{N}_{\mathbb{Y}}[\mathcal{T}(y_k) - \mathcal{T}(y_m)] < \alpha$, $\forall k, m \geq N$. Thus, $(\mathcal{T}(y_k))$ is a convex Cauchy sequence in $\mathbb{Y}$.

References

[1] J. R. Kider, New fuzzy normed spaces, *Baghdad Science Journal*, **9,** (2012), 559–564.

[2] J. Kider, M. Gheeab, Properties of a General Fuzzy Normed Space, *Iraqi Journal of Science*, **60,** no. 4, (2019), 847–855.

[3] Z.A. Khudhair, J.R. Kider, Some Properties of Fuzzy Compact Algebra Fuzzy Normed Spaces and Finite Dimensional Algebra Fuzzy Normed Spaces, *IOP Publishing, Journal of Physics: Conference Series*, 1879, (2021), 1–11.

[4] A.A. Khalafa, J.R. Kider, Linear operator of various types and its basic properties, *International Journal of Nonlinear Analysis and Applications*, **13,** no.1, (2022), 3949–3957.

[5] Hawraa Yousif Daher, Jehad R. Kider, Some properties of convex fuzzy normed space, *Int. J. Math. Comput. Sci.*, **19,** no. 2, (2024), 345-355.

[6] J. Kider, The convex fuzzy metric space, *AIP Conf. Proc.*, 3097, (2024), 080016.

[7] Eidi, E. Hameed, J. Kider, Convex fuzzy distance between two convex fuzzy compact sets, **27,** no. 4, (2024), 953–963.