International Journal of Mathematics and Computer Science
Volume **20,** Issue no. 1, (2025), 307–312
DOI: https://doi.org/10.69793/ijmcs/01.2025/rizky

Faithful Representations of Semigroup Crossed Products by Endomorphisms

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(Received September 22, 2024, Accepted October 22, 2024,
Published October 24, 2024)

Abstract

Given a cyclically ordered free Abelian group G, a unital C*-algebra A, and a semigroup homomorphism α from $P(G)$ into the semigroup $\mathrm{Endo}(A)$, we characterize faithful representations of crossed product $A \times_\alpha P(G)$ by endomorphisms.

1 Introduction

Hutington [2] introduced the cyclic order which is more general than the usual linear order. The cyclic order is a ternary relation on a set S. It is a subset R of S^3 and is written $R(u, v, w)$ when $(u, v, w) \in R$. The relation R is strict, cyclic, and transitive. A set G with the cyclic order R is called a cyclically ordered set. The cyclic order R is total if $u, v, w \in G$ such that $u \neq v \neq w \neq u$ implies $R(u, v, w)$ or $R(u, w, v)$. When G is a group, G is called cyclically ordered if R is compatible with the operation in G; i.e., if $R(u, v, w)$, then $R(a + u + b, a + v + b, a + w + b)$.

The positive cone of a cyclically ordered group $(G, +, R)$ is given by

$$P(G) := \{u \in G|\ R(0, u, 2u)\} \cup \{0\}.$$

Key words and phrases: Semigroup crossed product, endomorphism, free Abelian group, cyclically ordered group.
AMS (MOS) Subject Classifications: 06F15, 46L05.
ISSN 1814-0432, 2025, https://future-in-tech.net

The property of $P(G)$ is different with a positive cone of a linearly ordered group; i.e., in general, $P(G)$ is not necessarily a semigroup. For instance, the multiplicative group of complex numbers $\mathbb{T}$ of modulus one is cyclically ordered but $P(\mathbb{T})$ fails to be a semigroup. For any cyclically ordered group G in which the positive cone $P(G)$ is a semigroup, there is a representation [8] of $P(G)$ called an isometric representation of the semigroup $P(G)$ on the Hilbert space $\ell^2(P(G), \mathbb{C})$.

Given a free Abelian group G, the finite dimensional free Abelian groups are cyclically ordered and the positive cones are semigroups [4]. Later, Rosjanuardi, Gozali, and Albania [5] showed that the same result holds for any free Abelian groups. Given a unital C*-algebra A, they introduced the notion of semigroup crossed product by endomorphisms of A and that there is a dynamical system which has no crossed product ([5], Remark 3.1). In addition, they showed that the existence of crossed products depends on existence of covariant representation [5, Theorem 3.1]. If a dynamical system $(A, P(G), \alpha)$ admits a non trivial covariant representation, then there is a unique crossed product for the system, denoted by $A \times_\alpha P(G)$. In this article, we seek a characterization of faithful representation of the crossed product.

2 Main Results: Semigroup Crossed Products and Their Faithful Representations

Given a free Abelian group G, Theorem 2.3 of [5] implies that G is cyclically ordered, and the positive cone $P(G)$ is a semigroup. Given a unital C*-algebra A, a semigroup homomorphism α from $P(G)$ into the semigroup Endo(A) consists of all endomorphisms of A. The semigroup homomorphism α is called an action of $P(G)$ by endomorphisms of A. The system $(A, P(G), \alpha)$ is called a *dynamical system.* A *covariant representation* of $(A, P(G), \alpha)$ is a pair (π, V) in which π is a nondegenerate representation of A and V is an isometric representation of $P(G)$, such that the covariance condition $\pi(\alpha_t(a)) = V(t)\pi(a)V(t)^*$ for $a \in A$ and $t \in P(G)$ is satisfied.

Definition 2.1. *[5, Definition 3.1] The crossed product of $(A, P(G), \alpha)$ is a triple $(B, i_A, i_{P(G)})$ of a C*algebra B with a nondegenerate homomorphism $i_A : A \to B$, a homomorphism $i_{P(G)} : P(G) \to \operatorname{Isom} B$ such that*

1) $i_A(\alpha_t(a)) = i_{P(G)}(t) i_A(a) i_{P(G)}(t)^*$ for $a \in A$ and $t \in P(G)$;

2) for any covariant representation (π, V) of $(A, P(G), \alpha)$, there is a non-degenerate representation $\pi \times V$ of B such that $(\pi \times V) \circ i_A = \pi$ and $(\pi \times V) \circ i_{P(G)} = V$;

3) B is generated by $\{i_A(a) : a \in A\}$ and $\{i_{P(G)}(x) : x \in P(G)\}$.

We show that $A \times_\alpha P(G)$ carries a dual action $\hat{\alpha}$ of the dual group $\hat{G}$.

Theorem 2.2. *Suppose $(A, P(G), \alpha)$ is a dynamical system. Then there is a group homomorphism $\hat{\alpha} : \hat{G} \to \mathrm{Auto}(A \times_\alpha P(G))$ such that $\hat{\alpha}_\gamma(i_A(a)) = i_A(a)$ and $\hat{\alpha}_\gamma(i_{P(G)}(x)) = \gamma(x) i_{P(G)}(x)$ for $\gamma \in \hat{G}, a \in A, x \in P(G)$. Moreover, $\hat{\alpha}$ is continuous in the sense that, if γ_n converges to γ pointwise in $\hat{G}$, then $\hat{\alpha}_{\gamma_n}$ converges to $\hat{\alpha}_\gamma$ pointwise in $A \times_\alpha P(G)$.*

Proof. For every $\gamma \in \hat{G}$, the triple $(A \times_\alpha P(G), i_A, \gamma i_{P(G)})$ satisfies properties 1), 2) and 3) of Definition 2.1. Therefore, by [5, Theorem 3.2], there is an isomorphism $\hat{\alpha}_\gamma$ of $A \times_\alpha P(G)$ onto $A \times_\alpha P(G)$ such that $\hat{\alpha}_\gamma(i_A(a)) = i_A(a)$ and $\hat{\alpha}_\gamma(i_{P(G)}(x)) = \gamma(x) i_{P(G)}(x)$ for $\gamma \in \hat{G}, a \in A, x \in P(G)$. The continuity is clear.

□

We want to describe faithful representations of the crossed product.

Lemma 2.3. *We consider the crossed product $(A \times_\alpha P(G), i_A, i_{P(G)})$. For every $z \in P(G)$, the following set is a C*-algebra*

$$B_z := \mathrm{span}\left\{ i_{P(G)}(x)^* i_A(a) i_{P(G)}(x)) : a \in A, x \in P(G) \text{ with } z - x \in P(G) \right\}.$$

Proof. For every $a, b \in A$ and $x, y \in P(G)$ with $z - x, z - y \in P(G)$, we have $i_{P(G)}(x)^* i_A(a) i_{P(G)}(x) i_{P(G)}(y)^* i_A(b) i_{P(G)}(y) =$

$$\begin{cases} i_{P(G)}(y)^* i_A(\alpha_{y-x}(a)\alpha_y(1)b) i_{P(G)}(y) & \text{if } y - x \in P(G), \\ i_{P(G)}(x)^* i_A(\alpha_x(1))\alpha_{x-y}(b)) i_{P(G)}(x) & \text{else,} \end{cases}$$

and $(i_{P(G)}(x)^* i_A(a) i_{P(G)}(x))^* = i_{P(G)}(x)^* i_A(a^*) i_{P(G)}(x)$, then B_z is a *-algebra.

If $(b_n) \in B_z$ is a convergent sequence $(b_n) \in B_z$, then $i_{P(G)}(z) b_n i_{P(G)}(z)^*$ is of the form $i_A(a_n)$ for some $a_n \in A$. The convergence of (b_n) implies the convergence of $(i_A(a_n))$. Hence, $(i_A(a_n))$ is a Cauchy sequence in the image $i_A(A) \subseteq A \times_\alpha P(G)$. Then, $i_A(a_n)$ converges to $i_A(a)$ for some $a \in A$, since $i_A(A)$ is closed. Now, $b_n = i_{P(G)}(z)^* i_{P(G)}(z) b_n i_{P(G)}(z)^* i_{P(G)}(z)$ converges to $i_{P(G)}(z)^* i_A(a) i_{P(G)}(z) \in B_z$. Hence, B_z is closed and therefore it is a C*-algebra. □

If A is a C^*-algebra and H is a locally compact group, then a theory of integrating functions $f \in C_c(G, A)$ with respect to Haar measure μ on H can be discussed [9, Lemma C3].

Lemma 2.4. *We consider the crossed product $A \times_\alpha P(G)$. Then the dual action $\hat{\alpha}$ induces a norm decreasing projection θ of $A \times_\alpha P(G)$ onto the fixed point algebra*

$$(A \times_\alpha P(G))^{\hat{\alpha}} := \left\{ b \in A \times_\alpha P(G) : \hat{\alpha}_\gamma(b) = b \textit{ for all } \gamma \in \hat{G} \right\}$$

characterised by $\theta(b) = \int_{\hat{G}} \hat{\alpha}_\gamma(b) d\mu(\gamma)$.

Proof. Consider the dual action $\hat{\alpha}$ given by Theorem 2.2 of the compact group $\hat{G}$ on $A \times_\alpha P(G)$ characterised by $\hat{\alpha}_\gamma(i_A(a)) = i_A(a)$ and

$$\hat{\alpha}_\gamma(i_{P(G)}(x)) = \gamma(x) i_{P(G)}(x) \text{ for } \gamma \in \hat{G}, a \in A, x \in P(G).$$

For every $b \in A \times_\alpha P(G)$, we define $\psi_b : \hat{G} \to A \times_\alpha P(G)$ by $\gamma \longmapsto \hat{\alpha}_\gamma(b)$. Since $\hat{\alpha}$ is continuous, so is ψ_b. Hence, for every $b \in A \times_\alpha P(G)$, $\psi_b \in C(\hat{G}, A \times_\alpha P(G))$. According to Lemma C.3 of [9], the Haar integral $\int_{\hat{G}} \psi_b(\gamma) d\mu(\gamma)$ is in $A \times_\alpha P(G)$. Therefore, we can define a mapping $\theta : A \times_\alpha P(G) \to A \times_\alpha P(G)$ by $\theta(b) = \int_{\hat{G}} \psi_b(\gamma) d\mu(\gamma) = \int_{\hat{G}} \hat{\alpha}_\gamma(b) d\mu(\gamma)$. Linearity of the Haar integral implies that θ is linear. The invariance property of the Haar measure shows that θ defines a norm decreasing projection of $A \times_\alpha P(G)$ onto the fixed point algebra

$$(A \times_\alpha P(G))^{\hat{\alpha}} := \left\{ b \in A \times_\alpha P(G) : \hat{\alpha}_\gamma(b) = b \text{ for all } \gamma \in \hat{G} \right\}.$$

□

We show that a faithful representation of $A \times_\alpha P(G)$ can be generated from a covariant representation of the dynamical system.

Theorem 2.5. *Let (π, V) be a covariant representation of a dynamical system $(A, P(G), \alpha)$ such that*

(i) π is injective, and

(ii) for all finite subsets F of $P(G)$ and all choices $a_{x,y} \in A$,

$$\| \sum_{x \in F} V_x^* \pi(a_{x,x}) V_x \| \leq \| \sum_{x,y \in F} V_x^* \pi(a_{x,y}) V_y \|.$$

Then $\pi \times V$ is a faithful representation of $A \times_\alpha P(G)$.

Proof. Let (π, V) be a covariant representation of $(A, P(G), \alpha)$ satisfying the hypotheses. Lemma 2.4 gives a norm decreasing projection $\theta : A \times_\alpha P(G) \to (A \times_\alpha P(G))^{\hat{\alpha}}$ characterised by $\theta(b) = \int_{\hat{G}} \hat{\alpha}_\gamma(b) d\mu(\gamma)$. Moreover, θ is injective in the sense that if $\theta(b^*b) = 0$ for some $b \in A \times_\alpha P(G)$, then $b = 0$.

Since $\int_{\hat{G}} \gamma(y - x) d\mu(\gamma) = \begin{cases} 1 \text{ if } y=x \\ 0 \text{ else} \end{cases}$, linearity of θ and the integral implies that

$$\theta(\sum_{x,y} i_{P(G)}(x)^* i_A(a_{x,y}) i_{P(G)}(y)) = \sum_x i_{P(G)}(x)^* i_A(a_{x,x}) i_{P(G)}(x). \tag{2.1}$$

Therefore, $\pi \times V(\theta(\sum_{x,y} i_{P(G)}(x)^* i_A(a_{x,y}) i_{P(G)}(y))) = \sum_x V_x^* \pi(a_{x,x}) V_x$. Hence, the inequality (ii) in the hypotheses extends to:

$$\|\pi \times V(\theta(b))\| \leq \|\pi \times V(b)\|, \quad \text{for all } b \in A \times_\alpha P(G).$$

The proof is sufficient by showing that $\pi \times V$ is injective in the fixed point algebra $(A \times_\alpha P(G))^{\hat{\alpha}} = \theta(A \times_\alpha P(G))$. Continuity of θ and (2.1) imply that

$$(A \times_\alpha P(G))^{\hat{\alpha}} = \overline{\text{span}} \left\{ i_{P(G)}(x)^* i_A(a) i_{P(G)}(x) : a \in A, x \in P(G) \right\}.$$

If $b = \sum_x \lambda_x i_{P(G)}(x)^* i_A(a_x) i_{P(G)}(x)$ is in the spanning set satisfying $\pi \times V(b) = 0$, then $0 = \pi \times V(\sum_x \lambda_x i_{P(G)}(x)^* i_A(a_x) i_{P(G)}(x)) = \sum_x \lambda_x V_x^* \pi(a_x) V_x$. Now, choose x_0 be the element such that $x_0 - x \in P(G)$ for every x for which $a_x \neq 0$. Then, $0 = V_{x_0}(\sum_x \lambda_x V_x^* \pi(a_x) V_x) V_{x_0}^* = \pi(\sum_x \lambda_x \alpha_{x_0 - x}(\alpha_x(1) a_x \alpha_x(1)))$. Therefore, $i_A(\sum_x \lambda_x \alpha_{x_0 - x}(\alpha_x(1) a_x \alpha_x(1))) = 0$ which implies that $b = 0$. Thus, $\pi \times V$ is injective in span $\left\{ i_{P(G)}(x)^* i_A(a) i_{P(G)}(x) : a \in A, x \in P(G) \right\}$.

Since $\cup_{z \in P(G)} B_z = \text{ span} \left\{ i_{P(G)}(x)^* i_A(a) i_{P(G)}(x) : a \in A, x \in P(G) \right\}$, we get $\cup_{z \in P(G)} B_z$ is dense in $(A \times_\alpha P(G))^{\hat{\alpha}}$. Therefore, we obtain a family of C*-algebra $\{B_z\}$ such that $\overline{\cup_z B_z} = (A \times_\alpha P(G))^{\hat{\alpha}}$. By Lemma 1.3 of [1], we conclude that $\ker(\pi \times V) = \overline{\cup_z \ker(\pi \times V) \cap B_z}$. Since $\pi \times V$ is injective in $\cup_z B_z = \text{span} \left\{ i_{P(G)}(x)^* i_A(a) i_{P(G)}(x)) : a \in A, x \in P(G) \right\}$, we have

$$\cup_z (\ker(\pi \times V) \cap B_z) = \ker(\pi \times V) \cap (U_z B_z) = \{0\} \cap (\cup_z B_z) = \{0\}.$$

Thus, $\ker(\pi \times V) = \overline{\cup_z \ker(\pi \times V) \cap B_z} = \overline{\{0\}} = \{0\}$. Therefore, $\pi \times V$ is injective in the closure. $\square$

Acknowledgment. The research is supported by DRTPM Ditjen Dikti Kemendikbudristek Tahun Anggaran 2024.

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