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# Convex Neighborhood Polynomial of Generalized Wheel

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**Abstract**

In this paper, we characterize the convex subsets of the generalized wheel and establish its convex neighborhood polynomial.

## 1 Introduction

Graph polynomials have captured the attention of applied mathematicians because of the applications of these polynomials in other fields of sciences such as Chemistry, Biology, and Physics [4]. In 2014, Artes and Laja [**?**] introduced a pioneering work on convex subgraph polynomials which represents the number of convex subgraphs in a given graph structure. In 2022, Artes, Langamin, and Calib-og [3] introduced a bivariate graph polynomial called the clique common neighborhood polynomial of a graph by considering

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the common neighborhood system of a clique in a graph. In 2023, Arriesgado, Salim and Artes Jr. [2] investigated the clique connected common neighborhood polynomial of the join of graphs.

A subset $S$ of $V(G)$ is said to be *convex* if for every pair of vertices $\{u,v\} \subseteq S$, the set $I_G[u,v] = \{u,v\} \cup \{y : y \text{ lies in a } u\text{-}v \text{ path in } G\}$ is contained entirely in $S$. The *convex neighborhood polynomial* of a graph $G$ of order $n$ is given by $\Gamma_{cn}(G;x,y) = \sum_{j=0}^{n-i}\sum_{i=1}^{n} c_{ij}(G)x^i y^j$, where $c_{ij}(G)$ is the number of convex subsets in $G$ of cardinality $i$ with the corresponding neighborhood system of cardinality equal to $j$. This concept was first introduced by Abdurasid et al. [1] in 2023.

# 2 Results

We will use the following result of Laja [5] to characterize the convex sets in the generalized wheel $W_{m,n} = \overline{K_m} \oplus C_n$.

**Theorem 2.1.** [5] *Let $G_1$ and $G_2$ be noncomplete graphs. Then a proper subset $S = S_1 \cup S_2$ of $V(G_1 \oplus G_2)$, where $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$, is a convex set in $G_1 \oplus G_2$ if and only if $S_1$ and $S_2$ induce complete subgraphs in $G_1$ and $G_2$, respectively, where it may occur that $S_1 = \varnothing$ or $S_2 = \varnothing$.*

The following result characterizes the convex sets in the generalized wheel.

**Lemma 2.2.** *Let $m \geq 2$ and $n \geq 4$. A proper subset $S$ of $V(W_{m,n})$ is convex in $W_{m,n}$ if and only if it satisfies one of the following conditions.*

*(i)* *$S$ is a singleton subset of $V(W_{m,n})$.*

*(ii)* *$S$ induces a $K_2$ in $C_n$*

*(iii)* *$S = S_1 \cup S_2$, where $S_1$ is a singleton subset of $V(\overline{K_m})$ and $S_2$ is a singleton subset of $V(C_n)$.*

*(iv)* *$S = S_1 \cup S_2$, where $S_1$ is a singleton subset of $V(\overline{K_m})$ and $S_2$ induces a $K_2$ in $C_n$.*

*Proof.* Let $W_{m,n} = \overline{K_m} \oplus C_n$, where $V(\overline{K_m}) = \{u_1, u_2, \ldots, u_m\}$ and $C_n = [v_1, v_2, \ldots, v_n, v_1]$. Let $S$ be a convex subset of $V(W_{m,n})$. By Theorem 2.1, $S = S_1 \cup S_2$, where $S_1 \subseteq V(\overline{K_m})$ and $S_2 \subseteq V(C_n)$ with $S_1$ and $S_2$ induce complete subgraphs in $\overline{K_m}$ and $C_n$, respectively, where it may occur that $S_1 = \varnothing$ or $S_2 = \varnothing$.

Note that $S$ cannot contain more than one element from $V(\overline{K_m})$. Hence, if $S_2 = \varnothing$, then $S = S_1$ must be a singleton in $\overline{K_m}$. Note that the maximum order of complete subgraph in $C_n$ is 2. Hence, if $S_1 = \varnothing$, then $S = S_2$ is either a singleton or induces a $K_2$ in $C_n$. Hence, $(i)$ and $(ii)$ are satisfied.

Suppose $S_1 \neq \varnothing$ and $S_2 \neq \varnothing$. Then $S_1$ is a singleton subset of $V(\overline{K_m})$ and $S_2$ is a singleton subset of $V(C_n)$ or $S_2$ induces a $K_2$ in $C_n$, by Theorem 2.1. Thus, $(iii)$ and $(iv)$ are satisfied.

The converse is clear. □

The following result establishes the convex neighborhood polynomial of the generalized wheel $W_{m,n}$.

**Theorem 2.3.** *Let $m \geq 2$ and $n \geq 4$. Then*

$$\begin{aligned}\Gamma_{cn}(W_{m,n}; x, y) &= mxy^n + nxy^{m+2} + nx^2y^{m+2} + mnx^2y^{m+n-2}\\ &\quad + mnx^3y^{m+n-3} + x^{m+n}.\end{aligned}$$

*Proof.* Let $W_{m,n} = \overline{K_m} \oplus C_n$, where $V(\overline{K_m}) = \{u_1, u_2, \ldots, u_m\}$ and $C_n = [v_1, v_2, \ldots, v_n, v_1]$. Let $S$ be a proper convex subset of $W_{m,n}$. Then, by Lemma 2.2, $S$ satisfies one of the conditions listed in the lemma.

*Case 1.* $S$ is a singleton in $W_{m,n}$.

If $S = \{u\} \subset V(\overline{K_m})$., then $\deg_{W_{m,n}}(u) = n$. This contributes $mxy^n$ to the convex neighborhood of $W_{m,n}$, which gives the first term. If $S = \{v\} \subseteq V(C_n)$, then $\deg_{W_{m,n}}(v) = 2 + m$ since $v$ is adjacent to every vertex of $\overline{K_m}$ in $W_{m,n}$. This contributes $nxy^{m+2}$ to the convex neighborhood polynomial of $W_{m,n}$, which gives the second term.

*Case 2.* $S$ induces a $K_2$ in $C_n$.

If $S = \{v_i, v_{i+1}\}$, then $N_{W_{m,n}}(S) = V(\overline{K_m}) \cup \{v_{i-1}, v_{i+2}\}$ for $i \in \{1, 2, \ldots, n\}$, where $v_{n+1} = v_1$ and $v_{n+2} = v_2$. Hence, $|N_{W_{m,n}}(S)| = m+2$. This contributes $nx^2y^{m+2}$ to the convex neighborhood polynomial of $W_{m,n}$, which gives the third term.

*Case 3.* $S = S_1 \cup S_2$, where $S_1$ is a singleton subset of $V(\overline{K_m})$ and $S_2$ is a singleton subset of $V(C_n)$.

For each $i \in \{1, 2, \ldots, n\}$, let $S = \{u, v_i\}$, where $u \in V(\overline{K_m})$. Note that $N_{W_{m,n}}(u) = V(C_n)$ and $V(\overline{K_m}) \subset N_{W_{m,n}}(v_i)$. Hence, $N_{W_{m,n}}(S) = V(W_{m,n}) \setminus \{u, v_i\}$. Thus, $|N_{W_{m,n}}(S)| = m + n - 2$. This contributes $mnx^2y^{m+n-2}$ to the convex neighborhood polynomial of $W_{m,n}$, which gives the fourth term.

*Case 4.* $S = S_1 \cup S_2$, where $S_1$ is a singleton subset of $V(\overline{K_m})$ and $S_2$ induces a $K_2$ in $C_n$.

Let $S = \{u, v_i, v_{i+1}\}$, where $u \in V(\overline{K_m})$ and $i \in \{1, 2, \ldots, n\}$, where $v_{n+1} = v_1$. This set induces a triangle in $W_{m,n}$. Moreover, $N_{W_{m,n}}(u) = V(C_n)$ and for $i \in \{1, 2, \ldots, n\}$, $V(\overline{K_m}) \subset N_{W_{m,n}}(v_i)$. Hence, $|N_{W_{m,n}}(S)| = m + n - 3$. Note that there are $mn$ of these triangles in $W_{m,n}$. This contributes $mnx^3y^{m+n-3}$ to the convex neighborhood polynomial of $W_{m,n}$, which gives the fifth term.

The last term corresponds to the entire vertex set of $W_{m,n}$ which is a convex and empty neighborhood system. □

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