Fixed point theorem using contraction in complete multiplicative metric space

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Abstract

In our paper, we prove a fixed point theorem which extends results of Gupta and Garg and broadens a number of findings in the setting of multiplicative metric space.

1 Introduction

Fixed point theory holds importance across a range of disciplines including topology, mathematical economics, game theory, and approximation theory. A metric space is generally a non-empty abstract space with a distance function. In 1922, Banach established a standard result named ‘Banach Contraction Principle’, which is broadly regarded as the main source of “Metric Fixed Point Theory”. Then, several fixed point theorems were proved using this contraction principle. A new type of metric space called Multiplicative Metric Space (MMS) was coined by Bashirov et al. [1]. Various topological conditions in a multiplicative metric space were proved in [2]. The presented theorem extends the research results of Gupta and Garg [3] and broadens a number of findings in the setting of multiplicative metric space.

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number of well-known findings in the setting of multiplicative metric space.
Now, we recall the definition and topological conditions of MMS, which are required to prove our main result.

2 Preliminaries

Definition 2.1. [1] Let $\Xi$ be a non-empty set. Then a mapping $\varphi : \Xi \times \Xi \to [1, \infty)$ is called as multiplicative metric if the following conditions hold:

(i) $\varphi(\kappa, \omega) \geq 1$ for all $\kappa, \omega \in \Xi$.

(ii) $\varphi(\kappa, \omega) = \varphi(\omega, \kappa)$ for all $\kappa, \omega \in \Xi$.

(iii) $\varphi(\kappa, \gamma) \leq \varphi(\kappa, \omega) \cdot \varphi(\omega, \gamma)$ for all $\kappa, \omega, \gamma \in \Xi$.

Thus, $(\Xi, \varphi)$ is called a multiplicative metric space (MMS).

Lemma 2.2. [2] Let $(\Xi, \varphi)$ be MMS. If $\{\zeta_n\}$ is a sequence in $\Xi$ with $\kappa \in \Xi$, then $\{\zeta_n\} \to \kappa \ (n \to \infty) \Leftrightarrow \varphi(\zeta_n, \kappa) \to 1$ as $n \to \infty$.

Lemma 2.3. [2] Suppose $\{\zeta_n\}$ is a sequence in MMS $(\Xi, \varphi)$ with $\zeta \in \Xi$. We call $\{\zeta_n\}$ a multiplicative Cauchy sequence if and only if $\varphi(\zeta_n, \zeta_m) \to 1$ as $n, m \to \infty$.

Definition 2.4. [2] Let $(\Xi, \varphi)$ be MMS. We say $(\Xi, \varphi)$ is complete if every multiplicative Cauchy sequence in it is multiplicative convergent to $\zeta \in \Xi$.

3 Main results

Now, we prove our main theorem in which we apply the rational type contraction mapping in a complete MMS.

Theorem 3.1. Let $(\Xi, \varphi)$ be a complete MMS. Suppose $M : \Xi \to \Xi$ is continuous self mapping such that

\[
\varphi(M\zeta, M\tau) \leq [\varphi(\zeta, \tau)]^{\beta_1} \cdot [\varphi(\zeta, M\zeta) \cdot \varphi(\tau, M\tau)]^{\beta_2} \cdot [\varphi(\zeta, M\tau) \cdot \varphi(\tau, M\zeta)]^{\beta_3} \\
\cdot \left[\frac{\varphi(\zeta, M\tau)}{\varphi(\tau, M\tau)}\right]^{\beta_4} \cdot \left[\frac{\varphi(\zeta, M\tau)}{\varphi(\tau, \zeta)}\right]^{\beta_5}
\]

(3.1)

Also, $\forall \zeta, \tau \in \Xi, \zeta \neq \tau, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in [0, 1)$ such that $\beta_1 + 2\beta_2 + 2\beta_3 + \beta_4 + \beta_5 < 1$ and $\beta_1 + 2\beta_3 < 1$, $M$ has a unique fixed point in $\Xi$. 
Fixed point theorem using contraction...

Proof. Let \( \{ \zeta_n \} \) be a sequence in \( \Xi \) which, for \( \zeta_0 \in \Xi \), is defined as
\( M\zeta_n = \zeta_{n+1}, \forall n = 0, 1, 2, \ldots \)

\[
\varphi(\zeta_n, \zeta_{n+1}) = \varphi(M\zeta_{n-1}, M\zeta_n)
\]

\[
\leq [\varphi(\zeta_{n-1}, \zeta_n)]^{\beta_1} \cdot [\varphi(\zeta_{n-1}, M\zeta_n) \cdot \varphi(\zeta_n, M\zeta_n)]^{\beta_2} \cdot [\varphi(\zeta_{n-1}, M\zeta_n) \cdot \varphi(\zeta_n, M\zeta_{n-1})]^{\beta_3}
\]

\[
\leq [\varphi(\zeta_{n-1}, \zeta_n)]^{\beta_1} \cdot [\varphi(\zeta_{n-1}, \zeta_n) \cdot \varphi(\zeta_n, \zeta_{n+1})]^{\beta_2} \cdot [\varphi(\zeta_{n-1}, \zeta_n) \cdot \varphi(\zeta_n, \zeta_{n+1})]^{\beta_3}
\]

By continuing the process, we get
\[
\varphi(\zeta_{n-1}, \zeta_n) \leq [\varphi(\zeta_{n-2}, \zeta_n)]^{\beta_1} \cdot [\varphi(\zeta_{n-2}, \zeta_n) \cdot \varphi(\zeta_n, \zeta_{n+1})]^{\beta_2} \cdot [\varphi(\zeta_{n-2}, \zeta_n) \cdot \varphi(\zeta_n, \zeta_{n+1})]^{\beta_3}
\]

\[
[\varphi(\zeta_n, \zeta_{n+1})]^{(1-\beta_2-\beta_3-\beta_4)} \leq [\varphi(\zeta_{n-2}, \zeta_n)]^{(\beta_1+\beta_2+\beta_3+\beta_4)}
\]

Since \( \beta_1+2\beta_2+2\beta_3+\beta_4+\beta_5 < 1 \),
\[
[\varphi(\zeta_n, \zeta_{n+1})] \leq [\varphi(\zeta_{n-2}, \zeta_n)]^{k} \leq [\varphi(\zeta_{n-2}, \zeta_n)]^{k^2}.
\]

By continuing the process, we get \( \varphi(\zeta_n, \zeta_{n+1}) \leq [\varphi(\zeta_0, \zeta_1)]^{k^n} \).

Since \( 0 \leq k < 1 \), \( k^n \to 0 \) as \( n \to \infty \). Thus, \( \varphi(\zeta_n, \zeta_{n+1}) \to 1 \) and so \( \{ \zeta_n \} \) is a multiplicative Cauchy sequence.

By definition, \( \exists \) a point \( \zeta^* \in X \) such that \( \{ \zeta_n \} \to \zeta^* \).

As \( M \) is continuous, \( M(\zeta^*) = \lim_{n \to \infty} M(\zeta_n) = \lim_{n \to \infty} \zeta_{n+1} = \zeta^* \).

Hence, \( M \) has a fixed point. In order to demonstrate the uniqueness of \( M \), suppose \( \tau^* \) is another fixed point of \( M \). Then, by (3.1),
\[
\varphi(\zeta^*, \tau^*) = \varphi(M\zeta^*, M\tau^*)
\]

\[
\varphi(\zeta^*, \tau^*) \leq [\varphi(\zeta^*, \tau^*)]^{\beta_1} \cdot [\varphi(\zeta^*, M\zeta^*) \cdot \varphi(\tau^*, M\tau^*)]^{\beta_2} \cdot [\varphi(\zeta^*, M\tau^*) \cdot \varphi(\tau^*, M\zeta^*)]^{\beta_3}
\]

\[
\cdot \left[ \frac{\varphi(\zeta^*, M\zeta^*)}{\varphi(\tau^*, M\tau^*)} \right]^{\beta_4} \cdot \left[ \frac{\varphi(\zeta^*, M\tau^*)}{\varphi(\zeta^*, \tau^*)} \right]^{\beta_5}
\]
Since $\zeta^*$ and $\tau^*$ are fixed points,
\[
\varphi(\zeta^*, \tau^*) \leq \left[\varphi(\zeta^*, \zeta^*)\right]^{\beta_1} \cdot \left[\varphi(\zeta^*, \tau^*)\varphi(\tau^*, \zeta^*)\right]^{\beta_2} \cdot \left[\varphi(\zeta^*, \tau^*)\varphi(\tau^*, \zeta^*)\right]^{\beta_3}
\]
\[
\leq \left[\varphi(\zeta^*, \zeta^*)\right]^{\beta_1} \cdot \left[\varphi(\zeta^*, \tau^*)\right]^{\beta_2} \cdot \left[\varphi(\zeta^*, \tau^*)\right]^{\beta_3}
\]
\[
\varphi(\zeta^*, \tau^*) \leq \left[\varphi(\zeta^*, \zeta^*)\right]^{\beta_1+2\beta_3} \implies \left[\varphi(\zeta^*, \tau^*)\right]^{(1-\beta_1-2\beta_3)} \leq 1,
\]
a contradiction. Thus $\varphi(\zeta^*, \tau^*) = 1$. As a result, $\zeta^* = \tau^*$.

\[\Box\]

**Theorem 3.2.** Let $(\Xi, \varphi)$ be a complete MMS. Suppose $M : \Xi \to \Xi$ is a continuous self mapping such that
\[
\varphi(M\zeta, M\tau) \leq \left[\varphi(\zeta, \tau)\right]^{\beta_1} \cdot \left[\varphi(\zeta, M\zeta), \varphi(\tau, M\tau)\right]^{\beta_2} \cdot \left[\varphi(\zeta, \tau), \varphi(\tau, M\zeta)\right]^{\beta_3}
\]
\[
= \left[\varphi(\zeta, \tau)\right]^{\beta_1} \cdot \left(\varphi(\zeta, M\zeta), \varphi(\tau, M\tau)\right)\left[\varphi(\zeta, \tau), \varphi(\tau, M\zeta)\right]^{\beta_3}
\]
(3.2)

Also, $\forall \zeta, \tau \in \Xi, \zeta \neq \tau, \beta_1, \beta_2, \beta_3 \in [0, 1)$ such that $\beta_1 + 2\beta_2 + 4\beta_3 < 1$ and $\beta_1 + 2\beta_3 < 1$, $M$ possess unique fixed point in $\Xi$.

**Proof.** Let $\{\zeta_n\}$ be arbitrary sequence in $\Xi$.

Define $\zeta_0 \in \Xi$, such that $M\zeta_n = \zeta_{n+1}, \forall n = 0, 1, 2, ...$

Applying (3.2),
\[
\varphi(\zeta_n, \zeta_{n+1}) \leq \left[\varphi(\zeta_{n-1}, \zeta_n)\right]^{\left[\frac{\beta_1+2\beta_2+4\beta_3}{1-2\beta_2-2\beta_3}\right]}.
\]
\[
\varphi(\zeta_n, \zeta_{n+1}) \leq \left[\varphi(\zeta_{n-1}, \zeta_n)\right]^r, \text{ where } r = \frac{\beta_1+2\beta_2+4\beta_3}{1-2\beta_2-2\beta_3} < 1.
\]
Since $\beta_1 + 2\beta_2 + 4\beta_3 < 1$, by repeating iteration, we have
\[
\varphi(\zeta_n, \zeta_{n+1}) \leq \left[\varphi(\zeta_0, \zeta_1)\right]^n. \text{ Since } 0 \leq r < 1, r^n \to 0 \text{ as } n \to \infty \text{ and so } \varphi(\zeta_n, \zeta_{n+1}) = 1.
\]
Thus, $\{\zeta_n\}$ is multiplicative Cauchy sequence.

Since $\Xi$ is complete, $\lim_{n \to \infty} \zeta_n = l$.

As $M$ is continuous, $M(l) = M\left(\lim_{n \to \infty} \zeta_n\right) = \lim_{n \to \infty} M\zeta_n = \lim_{n \to \infty} \zeta_{n+1} = l$.

Thus $l$ is the fixed point of $M$. To prove uniqueness, if $Mg = g$, then, using (3.2),
\[
\varphi(l, g) \leq \left[\varphi(l, g)\right]^{(\beta_1+2\beta_3)} \text{ which implies } \left[\varphi(l, g)\right]^{(1-\beta_1-2\beta_3)} < 1, \text{ a contradiction, since } (\beta_1 + 2\beta_3) < 1.
\]

\[\Box\]

The following example satisfies all the hypotheses of Theorems 3.1 and 3.2.

**Example 3.3.** Let $\Xi = \mathbb{R}$ with the metric $\varphi(\zeta, \tau) = e^{\left|\zeta-\tau\right|}$. Since $\Xi$ is complete under the usual metric, so it is for $\varphi$.

Let $M(\zeta) = \frac{1}{2}\zeta + \kappa$, $\kappa \neq 0$ be a continuous map from $\mathbb{R}$ onto itself.

Note that,
\[
\varphi(\zeta(\zeta)), \varphi(M(\tau)) = e^{\left|M(\zeta)-M(\tau)\right|} = e^{\left|\zeta-\tau\right|} = \left(e^{\left|\zeta-\tau\right|}\right)^{\frac{1}{2}} = (d(\zeta, \tau))^{\frac{1}{2}}.
\]
Assuming that $\beta_1 = 1/2$ and $\beta_2 = \beta_3 = \beta_4 = \beta_5 = 0$,

we obtain the unique fixed point $\zeta = 2\kappa$. 

References


