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Fixed point theorem using contraction in complete multiplicative metric space

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Abstract

In our paper, we prove a fixed point theorem which extends results of Gupta and Garg and broadens a number of findings in the setting of multiplicative metric space.

1 Introduction

Fixed point theory holds importance across a range of disciplines including topology, mathematical economics, game theory, and approximation theory. A metric space is generally a non-empty abstract space with a distance function. In 1922, Banach established a standard result named 'Banach Contraction Principle', which is broadly regarded as the main source of "Metric Fixed Point Theory". Then, several fixed point theorems were proved using this contraction principle. A new type of metric space called Multiplicative Metric Space(MMS) was coined by Bashirov et al. [1]. Various topological conditions in a multiplicative metric space were proved in [2]. The presented theorem extends the research results of Gupta and Garg [3] and broadens a

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number of well-known findings in the setting of multiplicative metric space. Now, we recall the definition and topological conditions of MMS, which are required to prove our main result.

2 Preliminaries

Definition 2.1. *[1] Let Ξ be a non-empty set. Then a mapping $\varphi : \Xi \times \Xi \to [1, \infty)$ is called as multiplicative metric if the following conditions hold:*

(i) $\varphi(\varkappa, \omega) \geq 1$ *for all* $\varkappa, \omega \in \Xi$.

(ii) $\varphi(\varkappa, \omega) = \varphi(\omega, \varkappa)$ *for all* $\varkappa, \omega \in \Xi$.

(iii) $\varphi(\varkappa, \gamma) \leq \varphi(\varkappa, \omega).\varphi(\omega, \gamma)$ *for all* $\varkappa, \omega, \gamma \in \Xi$.

Thus, (Ξ, φ) is called a multiplicative metric space(MMS).

Lemma 2.2. *[2] Let (Ξ, φ) be MMS. If $\{\zeta_n\}$ is a sequence in Ξ with $\varkappa \in \Xi$, then $\{\zeta_n\} \to \varkappa$ $(n \to \infty) \Leftrightarrow \varphi(\zeta_n, \varkappa) \to 1$ as $n \to \infty$.*

Lemma 2.3. *[2] Suppose $\{\zeta_n\}$ is a sequence in MMS (Ξ, φ) with $\zeta \in \Xi$. We call $\{\zeta_n\}$ a multiplicative Cauchy sequence if and only if $\varphi(\zeta_n, \zeta_m) \to 1$ as $n, m \to \infty$.*

Definition 2.4. *[2] Let (Ξ, φ) be MMS. We say (Ξ, φ) is complete if every multiplicative Cauchy sequence in it is multiplicative convergent to $\zeta \in \Xi$.*

3 Main results

Now, we prove our main theorem in which we apply the rational type contraction mapping in a complete MMS.

Theorem 3.1. *Let (Ξ, φ) be a complete MMS. Suppose $M : \Xi \to \Xi$ is continuous self mapping such that*

$$\begin{aligned} \varphi(M\zeta, M\tau) \leq \; & [\varphi(\zeta, \tau)]^{\beta_1}.[\varphi(\zeta, M\zeta).\varphi(\tau, M\tau)]^{\beta_2}.[\varphi(\zeta, M\tau).\varphi(\tau, M\zeta)]^{\beta_3} \\ & .\left[\frac{\varphi(\zeta, M\tau)}{\varphi(\tau, M\tau)}\right]^{\beta_4}.\left[\frac{\varphi(\zeta, M\tau)}{\varphi(\zeta, \tau)}\right]^{\beta_5} \end{aligned} \tag{3.1}$$

Also, $\forall \zeta, \tau \in \Xi, \zeta \neq \tau, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in [0, 1)$ such that $\beta_1 + 2\beta_2 + 2\beta_3 + \beta_4 + \beta_5 < 1$ and $\beta_1 + 2\beta_3 < 1$, M has a unique fixed point in Ξ.

Proof. Let $\{\zeta_n\}$ be a sequence in Ξ which, for $\zeta_0 \in \Xi$, is defined as $M\zeta_n = \zeta_{n+1}, \forall n = 0, 1, 2, \ldots$

$$\begin{aligned}
&\varphi(\zeta_n, \zeta_{n+1}) = \varphi(M\zeta_{n-1}, M\zeta_n)\\
\leq\ & [\varphi(\zeta_{n-1}, \zeta_n)]^{\beta_1}.[\varphi(\zeta_{n-1}, M\zeta_{n-1}).\varphi(\zeta_n, M\zeta_n)]^{\beta_2}.[\varphi(\zeta_{n-1}, M\zeta_n).\varphi(\zeta_n, M\zeta_{n-1})]^{\beta_3}\\
&.\left[\tfrac{\varphi(\zeta_{n-1},M\zeta_n)}{\varphi(\zeta_n,M\zeta_n)}\right]^{\beta_4}.\left[\tfrac{\varphi(\zeta_{n-1},M\zeta_n)}{\varphi(\zeta_{n-1},\zeta_n)}\right]^{\beta_5}\\
\leq\ & [\varphi(\zeta_{n-1}, \zeta_n)]^{\beta_1}.[\varphi(\zeta_{n-1}, \zeta_n).\varphi(\zeta_n, \zeta_{n+1})]^{\beta_2}.[\varphi(\zeta_{n-1}, \zeta_{n+1}).\varphi(\zeta_n, \zeta_n)]^{\beta_3}\\
&.\left[\tfrac{\varphi(\zeta_{n-1},\zeta_{n+1})}{\varphi(\zeta_n,\zeta_{n+1})}\right]^{\beta_4}.\left[\tfrac{\varphi(\zeta_{n-1},\zeta_{n+1})}{\varphi(\zeta_{n-1},\zeta_n)}\right]^{\beta_5}\\
\leq\ & [\varphi(\zeta_{n-1}, \zeta_n)]^{\beta_1}.[\varphi(\zeta_{n-1}, \zeta_n).\varphi(\zeta_n, \zeta_{n+1})]^{\beta_2}.[\varphi(\zeta_{n-1}, \zeta_{n+1})]^{\beta_3}\\
&.\left[\tfrac{\varphi(\zeta_{n-1},\zeta_{n+1})}{\varphi(\zeta_n,\zeta_{n+1})}\right]^{\beta_4}.\left[\tfrac{\varphi(\zeta_{n-1},\zeta_{n+1})}{\varphi(\zeta_{n-1},\zeta_n)}\right]^{\beta_5}\\
\leq\ & [\varphi(\zeta_{n-1}, \zeta_n)]^{\beta_1}.[\varphi(\zeta_{n-1}, \zeta_n).\varphi(\zeta_n, \zeta_{n+1})]^{\beta_2}.[\varphi(\zeta_{n-1}, \zeta_n).\varphi(\zeta_n, \zeta_{n+1})]^{\beta_3}\\
&.\left[\tfrac{\varphi(\zeta_{n-1},\zeta_n).\varphi(\zeta_n,\zeta_{n+1})}{\varphi(\zeta_n,\zeta_{n+1})}\right]^{\beta_4}.\left[\tfrac{\varphi(\zeta_{n-1},\zeta_n).\varphi(\zeta_n,\zeta_{n+1})}{\varphi(\zeta_{n-1},\zeta_n)}\right]^{\beta_5}\\
\leq\ & [\varphi(\zeta_{n-1}, \zeta_n)]^{\beta_1}.[\varphi(\zeta_{n-1}, \zeta_n).\varphi(\zeta_n, \zeta_{n+1})]^{\beta_2}.[\varphi(\zeta_{n-1}, \zeta_n).\varphi(\zeta_n, \zeta_{n+1})]^{\beta_3}\\
&.[\varphi(\zeta_{n-1}, \zeta_n)]^{\beta_4}.[\varphi(\zeta_n, \zeta_{n+1})]^{\beta_5}
\end{aligned}$$

$[\varphi(\zeta_n, \zeta_{n+1})]^{(1-\beta_2-\beta_3-\beta_5)} \leq [\varphi(\zeta_{n-1}, \zeta_n)]^{(\beta_1+\beta_2+\beta_3+\beta_4)}$
$\varphi(\zeta_n, \zeta_{n+1}) \leq [\varphi(\zeta_{n-1}, \zeta_n)]^{\left(\frac{\beta_1+\beta_2+\beta_3+\beta_4}{1-\beta_2-\beta_3-\beta_5}\right)}$
$\varphi(\zeta_n, \zeta_{n+1}) \leq [\varphi(\zeta_{n-1}, \zeta_n)]^k$ where $k = \dfrac{\beta_1 + \beta_2 + \beta_3 + \beta_4}{1 - \beta_2 - \beta_3 - \beta_5} < 1$
Since $\beta_1+2\beta_2+2\beta_3+\beta_4+\beta_5 < 1, [\varphi(\zeta_n, \zeta_{n+1})] \leq [\varphi(\zeta_{n-1}, \zeta_n)]^k \leq [\varphi(\zeta_{n-2}, \zeta_{n-1})]^{k^2}$.
By continuing the process, we get $\varphi(\zeta_n, \zeta_{n+1}) \leq [\varphi(\zeta_0, \zeta_1)]^{k^n}$.
Since $0 \leq k < 1$, $k^n \to 0$ as $n \to \infty$. Thus, $\varphi(\zeta_n, \zeta_{n+1}) \to 1$ and so $\{\zeta_n\}$ is a multiplicative Cauchy sequence.
By definition, $\exists$ a point $\zeta^* \in X$ such that $\{\zeta_n\} \to \zeta^*$.
As M is continuous, $M(\zeta^*) = \lim_{n\to\infty} M(\zeta_n) = \lim_{n\to\infty} \zeta_{n+1} = \zeta^*$.
Hence, M has a fixed point. In order to demonstrate the uniqueness of M, suppose τ^* is another fixed point of M. Then, by (3.1),
$\varphi(\zeta^*, \tau^*) = \varphi(M\zeta^*, M\tau^*)$

$$\begin{aligned}
\varphi(\zeta^*, \tau^*) \leq\ & [\varphi(\zeta^*, \tau^*)]^{\beta_1}.[\varphi(\zeta^*, M\zeta^*).\varphi(\tau^*, M\tau^*)]^{\beta_2}.[\varphi(\zeta^*, M\tau^*).\varphi(\tau^*, M\zeta^*)]^{\beta_3}\\
&.\left[\tfrac{\varphi(\zeta^*,M\zeta^*)}{\varphi(\tau^*,M\tau^*)}\right]^{\beta_4}.\left[\tfrac{\varphi(\zeta^*,M\tau^*)}{\varphi(\zeta^*,\tau^*)}\right]^{\beta_5}
\end{aligned}$$

Since ζ^* and τ^* are fixed points,

$$\begin{aligned}\varphi(\zeta^*,\tau^*) \leq\ & [\varphi(\zeta^*,\tau^*)]^{\beta_1}.[\varphi(\zeta^*,\zeta^*).\varphi(\tau^*,\tau^*)]^{\beta_2}.[\varphi(\zeta^*,\tau^*).\varphi(\tau^*,\zeta^*)]^{\beta_3}\\ & .\left[\tfrac{\varphi(\zeta^*,\zeta^*)}{\varphi(\tau^*,\tau^*)}\right]^{\beta_4}.\left[\tfrac{\varphi(\zeta^*,\tau^*)}{\varphi(\zeta^*,\tau^*)}\right]^{\beta_5}\\ \varphi(\zeta^*,\tau^*) \leq\ & [\varphi(\zeta^*,\tau^*)]^{\beta_1+2\beta_3} \implies [\varphi(\zeta^*,\tau^*)]^{(1-\beta_1-2\beta_3)} \leq 1,\end{aligned}$$

a contradiction. Thus $\varphi(\zeta^*,\tau^*) = 1$. As a result, $\zeta^* = \tau^*$.

□

Theorem 3.2. *Let (Ξ,φ) be a complete MMS. Suppose $M : \Xi \to \Xi$ is a continuous self mapping such that*

$$\begin{aligned}\varphi(M\zeta, M\tau) \leq\ & [\varphi(\zeta,\tau)]^{\beta_1}.\left[(\varphi(\zeta,M\zeta).\varphi(\tau,M\tau)).\left(\frac{\varphi(\zeta,\tau).\varphi(\tau,M\tau)}{\varphi(\zeta,M\tau)}\right)\right]^{\beta_2}\\ & .[(\varphi(\zeta,M\tau).\varphi(\tau,M\zeta)).(\varphi(\zeta,\tau).\varphi(\tau,M\tau))]^{\beta_3}\end{aligned} \tag{3.2}$$

Also, $\forall \zeta,\tau \in \Xi, \zeta \neq \tau, \beta_1,\beta_2,\beta_3 \in [0,1)$ such that $\beta_1 + 2\beta_2 + 4\beta_3 < 1$ and $\beta_1 + 2\beta_3 < 1$, M possess unique fixed point in Ξ.

Proof. Let $\{\zeta_n\}$ be arbitrary sequence in Ξ.
Define $\zeta_0 \in \Xi$, such that $M\zeta_n = \zeta_{n+1}, \forall n = 0,1,2,...$
Applying (3.2), $\varphi(\zeta_n,\zeta_{n+1}) \leq [\varphi(\zeta_{n-1},\zeta_n)]^{\left[\frac{\beta_1+\beta_2+2\beta_3}{1-\beta_2-2\beta_3}\right]}$
$\varphi(\zeta_n,\zeta_{n+1}) \leq [\varphi(\zeta_{n-1},\zeta_n)]^r$, where $r = \frac{\beta_1+\beta_2+2\beta_3}{1-\beta_2-2\beta_3} < 1$.
Since $\beta_1 + 2\beta_2 + 4\beta_3 < 1$, by repeating iteration, we have
$\varphi(\zeta_n,\zeta_{n+1}) \leq [\varphi(\zeta_0,\zeta_1)]^{r^n}$. Since $0 \leq r < 1$, $r^n \to 0$ as $n \to \infty$ and so $\varphi(\zeta_n,\zeta_{n+1}) = 1$. Thus, $\{\zeta_n\}$ is multiplicative Cauchy sequence.
Since Ξ is complete, $\lim\limits_{n\to\infty} \zeta_n = l$.
As M is continuous, $M(l) = M\left(\lim\limits_{n\to\infty}\zeta_n\right) = \lim\limits_{n\to\infty} M\zeta_n = \lim\limits_{n\to\infty}\zeta_{n+1} = l$.
Thus l is the fixed point of M. To prove uniqueness, if $Mg = g$, then, using (3.2), $\varphi(l,g) \leq [\varphi(l,g)]^{(\beta_1+2\beta_3)}$ which implies $[\varphi(l,g)]^{(1-\beta_1-2\beta_3)} < 1$, a contradiction, since $(\beta_1 + 2\beta_3) < 1$. □

The following example satisfies all the hypotheses of Theorems 3.1 and 3.2.

Example 3.3. *Let $\Xi = \mathbb{R}$ with the metric $\varphi(\zeta,\tau) = e^{|\zeta-\tau|}$. Since Ξ is complete under the usual metric, so it is for φ.*
Let $M(\zeta) = \frac{1}{2}\zeta + \kappa,\ \kappa \neq 0$ be a continuous map from $\mathbb{R}$ onto itself.
Note that, $\varphi(M(\zeta),M(\tau)) = e^{|M(\zeta)-M(\tau)|} = e^{\frac{1}{2}|\zeta-\tau|} = (e^{|\zeta-\tau|})^{\frac{1}{2}} = (d(\zeta,\tau))^{\frac{1}{2}}$.
Assuming that $\beta_1 = 1/2$ and $\beta_2 = \beta_3 = \beta_4 = \beta_5 = 0$,
we obtain the unique fixed point $\zeta = 2\kappa$.

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