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On the Non-Reality of Totally Positive Fields

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Abstract

Let F be a formally real field and let F_{py} be the Pythagorean closure of F. We discuss totally positive field extensions and prove that a totally positive field extension need not be totally real. For a central simple algebra A over a field F, $\text{pind}(A)$ denotes the index of A over F_{py}. We also prove a special case of the Becher's conjecture.

1 Introduction

Let F be a formally real field, meaning that -1 cannot be expressed as a sum of squares in F. Recall that a field F is said to be Pythagorean if every sum of squares in F is a square.

A field extension K/F is considered *totally positive* if every semiordering defined on F can be extended to a semiordering on K (see §3, [1]). In this article we prove that any totally positive field extension need not be *totally real*. If A is a central simple algebra over F with index $\text{ind}\,(A)$, the index of $A \otimes_F F_{\text{py}}$ is called the Pythagorean index of A and is denoted by $\text{pind}(A)$.

The following conjecture, attributed to K. Becher, proposes conditions under which the Pythagorean index remains unchanged over a field extension K/F for the 2-torsion elements in the Brauer group of F.

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Conjecture 1.1 (Becher). *Let K/F be a totally positive field extension. For a central simple algebra A over F of exponent 2, one has* $\operatorname{pind}(A) = \operatorname{pind}(A \otimes_F K)$.

In this article, we prove Conjecture 1.1 for quaternion division algebras over F.

2 Preliminaries

It is assumed that all fields discussed have characteristics distinct from 2. A quadratic form q over F is termed *isotropic* if there exists a nonzero vector $v \in V$ such that $q(v) = 0$. For any $n \in \mathbb{N}$, the notation $n \cdot q$ denotes the n-fold orthogonal sum of q. Moreover, q is considered *weakly isotropic* if there exists $n \in \mathbb{N}$ such that $n \cdot q$ is isotropic.

Let $\sigma(F)$ be the collection of elements in F which are sum of squares in F. If -1 is not a sum of squares in F, we call F is formally real, i.e., F is formally real if $-1 \notin \sigma(F)$. A field F is called *Pythagorean* if $\sigma(F) \subset F$. The Pythagorean closure of F, denoted by F_{py}, is the smallest subfield of F_{al} containing F that is Pythagorean.
We next define the ordering and the semiordering on a field F

Definition 2.1. *An ordering P on F is a proper subset $P \subsetneq F$ such that $F^2 \subseteq P, P + P \subseteq P$, $P \cdot P \subseteq P$, $P \cup -P = F$, and $P \cap -P = \{0\}$.*

Definition 2.2. *A semiordering on a field F is a subset $S \subset F$ satisfying the following conditions:* $1 \in S$, $F^2 S \subset S$, $S + S \subset S$, $S \cup -S = F$, *and* $S \cap -S = \{0\}$.

Thus, every ordering on a field is a semiordering. But every semiordering need not be an ordering (see [3], for more details). A field extension K/F is said totally positive if every semiordering on F extends to a semiordering on K. Despite the challenge of identifying semiorderings, Becher provides a characterization of total positiveness using quadratic forms: A field extension K/F is totally positive if and only if for a quadratic form q defined over F that is isotropic over K then q must be weakly isotropic over F.

3 Main results

Recall that an algebraic number field, also known as a number field, is defined as a finite-degree field extension of $\mathbb{Q}$, where "degree" refers to the dimension of the field considered as a vector space over $\mathbb{Q}$.

Definition 3.1. *A finite field extension K of $\mathbb{Q}$ is said to be totally real if every field embedding of K in the complex numbers is contained in the real numbers.*

The following theorem is proved in [3].

Theorem 3.2. *Let K be a totally real number field. Then $K/\mathbb{Q}$ is a totally positive field extension.*

However, we show that the converse to the above theorem need not be true.

Theorem 3.3. *A totally positive extension of $\mathbb{Q}$ need not be totally real.*

Proof. For instance, consider the cubic polynomial $f(x) = x^3 - p$, where p is any prime number. Then f has one real root $\sqrt[3]{p}$ and two complex roots given by $\sqrt[3]{p}\omega$ and $\sqrt[3]{p}\omega^2$, where $\omega \in \mathbb{C}$ is a primitive 3^{rd} root of unity such that $\omega^3 = 1$. Then $\mathbb{Q}(\sqrt[3]{p})$ is a totally positive extension of $\mathbb{Q}$ as it has odd degree 3 (see Chapter 7, Theorem 2.7, [2]) but it is not totally real. □

The following theorem demonstrating that the Becher's conjecture 1.1 is valid for a quaternion division algebra over F.

Theorem 3.4. *Let F be a formally real field, and K/F be a totally positive field extension. Suppose that D is a quaternion division algebra over F. Then* $\mathrm{pind}(D) = \mathrm{pind}(D \otimes_F K)$.

Proof. If $\mathrm{pind}(D) = 1$, then we are done. Suppose $\mathrm{pind}(D) = 2$ and $\mathrm{pind}(D \otimes_F K) = 1$. Then the norm form n_D, of D over F is isotropic over K_{py} (see Theorem 2.7, Chapter 3, [2]). As K_{py}/K and K/F is totally positive, we have K_{py}/F is totally positive. Thus, n_D is weakly isotropic over F and hence weakly isotropic over F_{py}. As F_{py} is Pythagorean, n_D becomes isotropic over F_{py}. This is a contradiction as $\mathrm{pind}(D) = 2$. □

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