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On the Diophantine Equation $a^x + (a+5b)^y = z^2$

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Abstract

In this paper, we study the Diophantine equation $a^x+(a+5b)^y = z^2$ when $a \equiv 1 \pmod 5$ and b is a positive integer. We establish that the equation has no solutions in positive integers x, y and z. We start with the Diophantine equation $p^x + (p+5a)^y = z^2$ where p and $p+5a$ are both primes and $p \equiv 1 \pmod 5$ and a is a positive integer.

1 Introduction

Many researchers have studied Diophantine equations and continue to do so. In 1844, Catalan [5] conjectured that $(3, 2, 2, 3)$ is the unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = z^2$ where a, b, x and y are integers such that $\min\{a, b, x, y\} > 1$. The Catalan's conjecture was proved by Mihailescu [8] in 2004. Numerous researchers have studied the exponential Diophantine equation $a^x + b^y = z^2$.

In 2013, Chotchaisthit. [6] solved the Diophantine equation $p^x+(p+1)^y = z^2$ where p is a Mersenne prime. In 2018, Burshtein [1] showed that the Diophantine equation $p^x + (p+4)^y = z^2$ when $p, p+4$ are primes, has no solution (x, y, z) in positive integers. For the cases when there are solutions he exhibited them [2]. In the same year, Burshtein [3] also studied

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the Diophantine equation $p^x + (p+6)^y = z^2$ when $p, p+6$ are primes and $x+y = 2, 3, 4$. Still in 2018, Fernando [7] showed that the Diophantine equation $p^x + (p+8)^y = z^2$ when $p > 3$ and $p+8$ are primes has no solutions (x, y, z) in positive integers. In 2020, Burshtein [4] studied the Diophantine equation $p^x + (p+5)^y = z^2$ when $p+5 = 2^{2u}$. In this paper we study the Diophantine equation $a^x + (a+5b)^y = z^2$ when $a \equiv 1 \pmod{5}$ and b is a positive integer. We establish that the equation has no solutions in positive integers x, y and z. We start with the Diophantine equation $p^x + (p+5a)^y = z^2$ where p and $p+5a$ are both primes and $p \equiv 1 \pmod{5}$ and a is a positive integer.

2 Preliminaries

Lemma 2.1. *If $a \equiv 1 \pmod{5}$, then the Diophantine equation $a^x + 1 = z^2$ has no solutions where x and z are non-negative integers.*

Proof. If $x = 0$, then $2 = z^2$ which is impossible. If $x = 1$, then $a + 1 = z^2$. Since $a \equiv 1 \pmod{5}$, $z^2 = a + 1 \equiv 2 \pmod{5}$ which is impossible. If $x > 1$, then $a^x + 1 = z^2$ or equivalently $z^2 - a^x = 1$ which is impossible by Catalan's conjecture and $a \equiv 1 \pmod{5}$. □

Lemma 2.2. *If $a \equiv 1 \pmod{5}$, then the Diophantine equation $1 + (a+5b)^y = z^2$ where b is a positive integer, y and z are non-negative integers, has no solutions.*

Proof. If $y = 0$, then $2 = z^2$ which is impossible. If $y = 1$, then $1 + (a+5b) = z^2$. Since $a \equiv 1 \pmod{5}$, $a + 5b \equiv 1 \pmod{5}$ and so $z^2 = 1 + a + 5b \equiv 2 \pmod{5}$ which is impossible. If $y > 1$, then $1 + (a+5b)^y = z^2$ or equivalently $z^2 - (a+5b)^y = 1$ which is impossible by Catalan's conjecture and $a \equiv 1 \pmod{5}$. □

3 Main results

Theorem 3.1. *If p and $p+5a$ are both primes such that $p \equiv 1 \pmod{5}$ and a is a positive integer, then the Diophantine equation $p^x + (p+5a)^y = z^2$ has no solutions in non-negative integer x, y and z.*

Proof. If $x = 0$, then $1 + (p+5a)^x = z^2$ which has no solution by Lemma 2.2.
If $y = 0$, then $p^x + 1 = z^2$ which has no solution by Lemma 2.1.
If $x \geq 1$ and $y \geq 1$, then p^x and $(p+5a)^y$ both are odd. Thus z^2 is even.

Therefore, $z^2 \equiv 0 \pmod{5}$ or $z^2 \equiv 1 \pmod{5}$ or $z^2 \equiv 4 \pmod{5}$. Since $p \equiv 1 \pmod{5}$, $p^x \equiv 1 \pmod{5}$ and $(p+5a)^y \equiv 1 \pmod{5}$. Hence, $p^x + (p+5a)^y \equiv 2 \pmod{5}$ which is impossible. □

Theorem 3.2. *If $a \equiv 1 \pmod{5}$ and a is a positive integer, then the Diophantine equation $a^x + (a+5b)^y = z^2$ has no solutions where x, y and z are non-negative integers and b is a positive integer.*

Proof. We consider four cases.
Case I: For $a \equiv 1 \pmod{5}$ and b are odd numbers.
If $x = 0$, then $1 + (a+5b)^y = z^2$ which has no solution by Lemma 2.2.
If $y = 0$, then $a^x + 1 = z^2$ which has no solution by Lemma 2.1.
If $x \geq 1$ and $y \geq 1$, then a^x is odd and $(a+5b)^y$ is even. Thus z^2 is odd. Therefore, $z^2 \equiv 1 \pmod{5}$ or $z^2 \equiv 4 \pmod{5}$. Since a is odd and $a \equiv 1 \pmod{5}$, $a \equiv 1 \pmod{5}$ and so $a^x \equiv 1 \pmod{5}$ and $(a+5b)^y \equiv 1 \pmod{5}$. Hence, $a^x + (a+5b)^y \equiv 2 \pmod{5}$ which is impossible.

Case II: For $a \equiv 1 \pmod{5}$ is an odd number and b is an even.
If $x = 0$, then $1 + (a+5b)^y = z^2$ which has no solution by Lemma 2.2.
If $y = 0$, then $a^x + 1 = z^2$ which has no solution by Lemma 2.1.
If $x \geq 1$ and $y \geq 1$, then a^x and $(a+5b)^y$ both are odd. Thus z^2 is even. Therefore, $z^2 \equiv 0 \pmod{5}$ or $z^2 \equiv 1 \pmod{5}$ or $z^2 \equiv 4 \pmod{5}$. Since a is odd and $a \equiv 1 \pmod{5}$, $a^x \equiv 1 \pmod{5}$ and $(a+5b)^y \equiv 1 \pmod{5}$. Hence, $a^x + (a+5b)^y \equiv 2 \pmod{5}$ which is impossible.

Case III: For $a \equiv 1 \pmod{5}$ is an even number and b is an odd number.
If $x = 0$, then $1 + (a+5b)^y = z^2$ which has no solution by Lemma 2.2.
If $y = 0$, then $a^x + 1 = z^2$ which has no solution by Lemma 2.1.
If $x \geq 1$ and $y \geq 1$, then a^x is even and $(a+5b)^y$ is odd. Thus z^2 is odd. Therefore, $z^2 \equiv 1 \pmod{5}$ or $z^2 \equiv 4 \pmod{5}$. Since a is even and $a \equiv 1 \pmod{5}$, $a^x \equiv 1 \pmod{5}$ and $(a+5b)^y \equiv 1 \pmod{5}$. Hence, $a^x + (a+5b)^y \equiv 2 \pmod{5}$ which is impossible.

Case IV: For $a \equiv 1 \pmod{5}$ and b are even numbers.
If $x = 0$, then $1 + (a+5b)^y = z^2$ which has no solution by Lemma 2.2.
If $y = 0$, then $a^x + 1 = z^2$ which has no solution by Lemma 2.1.
If $x \geq 1$ and $y \geq 1$, then a^x and $(a+5b)^y$ are both even. Thus z^2 is even. Therefore, $z^2 \equiv 0 \pmod{5}$ or $z^2 \equiv 1 \pmod{5}$ or $z^2 \equiv 4 \pmod{5}$. Since a is even and $a \equiv 1 \pmod{5}$ so, $a^x \equiv 1 \pmod{5}$ and $(a+5b)^y \equiv 1 \pmod{5}$. Consequently, $a^x + (a+5b)^y \equiv 2 \pmod{5}$ which is impossible. □

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