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The Vertex Distance on Cayley Digraphs of Rectangular Groups with respect to the Cartesian Product Connection Sets

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Abstract

A rectangular group S is a semigroup isomorphic to the direct product of a group, a left zero semigroup, and a right zero semigroup. The Cayley digraph $Cay(S, A)$ of a finite rectangular group S with respect to a nonempty subset A of S is defined as a digraph with vertex set S and arc set consisting of ordered pairs $(u, ua) \in S \times S$ for some $a \in A$. The set A is called a connection set of $Cay(S, A)$. Let $u, v \in S$. The distance from u to v in $Cay(S, A)$, denoted by $d(u, v)$, is the number of arcs in the shortest directed $u - v$ path if one exists and is ∞ otherwise. In this paper, we characterize the distances of any two vertices in Cayley digraphs of rectangular groups with respect to the Cartesian product connection sets.

1 Introduction and Preliminaries

The origins of Algebraic graph theory can be traced back to the mid-20th century when mathematicians began exploring the connections between graphs and algebraic structures. Then the field has grown significantly and found

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many applications in various areas including computer science, chemistry, physics, and social sciences. Algebraic graph theory is a fascinating field, at the intersection of Algebra and Graph Theory, which studies the properties and structures of graphs using algebraic techniques. It provides a powerful framework for understanding graphs by representing them using algebraic structures such as groups, semigroups, rings, and matrices. One particular area of focus within this field is the construction of Cayley digraphs from rectangular groups. The relationship between Cayley digraphs and rectangular groups has been widely studied, as evidenced by the numerous references. In 2009, Panma et al. [9] presented the transitive Cayley graphs of strong semilattices of right (left) groups. In 2010, Khosravi and Mahmoudi [4] found Cayley graphs which are automorphism-vertex-transitive. In 2010, Panma [8] provided characterizations for Cayley graphs of rectangular groups. Later in 2016, Meksawang and Panma [5] investigated isomorphism conditions for Cayley graphs of rectangular groups. Furthermore, in 2018, Panma and Nupo [10] determined the independence number of Cayley digraphs of rectangular groups. In the same year, they [6] verified the independent domination number in Cayley digraphs of rectangular groups. Indeed, the study of vertex distance in graphs or digraphs corresponds to the lengths of paths or directed paths. We focus our research on the investigation of vertex distance and directed paths in Cayley digraphs of rectangular groups. Therefore, it will be important to consider those structural properties of Cayley digraphs. Recently, in 2023, Nupo and Panma [7] studied certain structural properties for Cayley regularity graphs of semigroups and their theoretical applications.

We now provide some crucial preliminaries and notations which are useful for this paper. Moreover, we refer to [3] for additional information on algebraic semigroups and to [1, 2, 11] for details regarding graphs and digraphs. The following definitions and theorem are beneficial for investigating the structural properties for Cayley digraphs of rectangular groups.

Definition 1.1. [3] A semigroup S is called a *left zero semigroup* if $xy = x$ and is called a *right zero semigroup* if $xy = y$ for all $x, y \in S$. In addition, the semigroup S is called a *rectangular group* if S is isomorphic to $G \times L \times R$, the direct product of a group G, a left zero semigroup L, and a right zero semigroup R. Moreover, the semigroups $G \times L$ and $G \times R$ are called a *left group* and a *right group*, respectively.

Definition 1.2. [11] A *digraph (directed graph)* D is a pair $(V(D), E(D))$ in which $V(D)$ is a nonempty set whose elements are called the *vertices* and $E(D)$ is the subset of the set of ordered pairs of elements in $V(D)$. The elements of $E(D)$ are called the *arcs* of D. The set $V(D)$ is called the *vertex set* of D and the set $E(D)$ is called the *arc set* of D.

Definition 1.3. [2] Let S be a semigroup and let A a nonempty subset of S. The *Cayley digraph* $Cay(S, A)$ of a *semigroup* S with respect to a *connection set* A is defined to be a digraph with the vertex set $V(Cay(S, A)) = S$ and the arc set $E(Cay(S, A)) = \{(x, xa) \in S \times S : x \in S \text{ and } a \in A\}$.

Definition 1.4. [1] Let D be a directed graph and $u, v \in V(D)$. Then the *distance from u to v* in D, denoted by $d(u, v)$, is the number of arcs in the shortest directed $u - v$ path if one exists and is ∞ if there is no directed path joining u to v in D.

Remark 1.5. In the area of directed graphs, we can observe that $d(u, v)$ may be distinct from $d(v, u)$ for any two vertices u, v of a digraph.

Theorem 1.6. *[6] Let $S \cong G \times L \times R$ be a rectangular group and let A a nonempty subset of S. If $\overline{A} = \{(a, \alpha) \in G \times R : (a, l, \alpha) \in A \text{ for some } l \in L\}$, then $Cay(S, A)$ is the disjoint union of $|L|$ strong subdigraphs which each subdigraph is isomorphic to the Cayley digraph $Cay(G \times R, \overline{A})$ of a right group $G \times R$ with a connection set $\overline{A}$.*

For convenience, we denote by Γ the Cayley digraph $Cay(S, A)$ of a rectangular group S with a connection set A. Hereafter, we focus on the rectangular group $\mathbb{Z}_n \times L \times R$ where $\mathbb{Z}_n$ is a finite cyclic group of order n under the addition whose elements are integers modulo n. Our goal is to characterize the vertex distance $d(u, v)$ in Γ for any two distinct elements $u, v \in V(\Gamma)$. In particular, a connection set A of Γ considered in this paper is related to the Cartesian product provided that $p_1(A)$ is a cyclic subgroup of $\mathbb{Z}_n$ where $p_1(A)$ means the first projection of A. We denote all elements in $\mathbb{Z}_n$ by integers from 0 to $n - 1$ where $n \geq 2$. All sets mentioned in this paper are assumed to be finite.

2 The Vertex Distance on Γ

In order to prove the following theorem, it is necessary to define the notation $V_j = \langle a \rangle + j$ such that $0 \leq j < a$ and $\langle a \rangle$ is a cyclic subgroup of $\mathbb{Z}_n$ generated by $a \in \mathbb{Z}_n$. We first start with the case where $p_1(A)$ is a cyclic subgroup of $\mathbb{Z}_n$ excluding the zero element. Next, let $(a_1, l_1, r_1), (a_2, l_2, r_2) \in \mathbb{Z}_n \times L \times R$. It can be easily observed that if $r_2 \notin p_3(A)$, the third projection of A, then the in-neighborhood $N^-((a_2, l_2, r_2)) = \emptyset$ where the notation $N^-((a_2, l_2, r_2))$ means the set $\{(a, l, r) \in V(\Gamma) : ((a, l, r), (a_2, l_2, r_2)) \in E(\Gamma)\}$. It follows that $d((a_1, l_1, r_1), (a_2, l_2, r_2)) = \infty$.

Theorem 2.1. *Let n be a positive integer greater than or equal 2. Let Γ be the Cayley digraph of a rectangular group $\mathbb{Z}_n \times L \times R$ with respect to the connection set $A = V_0 \setminus \{0\} \times B \times C$ where $V_0 = \langle a \rangle$ is a cyclic subgroup of $\mathbb{Z}_n$, $B \subseteq L$ and $C \subseteq R$. For each $(c, l_i, r_\eta), (f, l_\theta, r_\beta) \in V(\Gamma)$,*

$$d((c, l_i, r_\eta), (f, l_\theta, r_\beta)) = \begin{cases} 1 & \text{if } f \neq c \text{ where } f, c \in V_x\,, i = \theta \text{ and } r_\beta \in C; \\ 2 & \text{if } f = c\,, i = \theta \text{ and } r_\beta \in C; \\ \infty & \text{if otherwise} \end{cases}$$

for some $x \in \{0, 1, 2, \ldots, a-1\}$.

Proof. Let $(c, l_i, r_\eta), (f, l_\theta, r_\beta) \in V(\Gamma)$ and $V_0 = \langle a \rangle = \{0, a, 2a, ..., (k-1)a\}$ be a cyclic subgroup of $\mathbb{Z}_n$ of order k generated by $a \in \mathbb{Z}_n$ for some $k \in \mathbb{N}$. For convenience, we can assume, without loss of generality, that $i = 1$ and let $b = (f, l_\theta, r_\beta)$. Consider the following cases.
Case 1 : $r_\beta \notin C$. Then there is no $(x, l, r) \in A$ in which $(c, l_1, r_\eta)(x, l, r) = (f, l_\theta, r_\beta)$ and hence $d((c, l_i, r_\eta), b) = d((c, l_1, r_\eta), (f, l_\theta, r_\beta)) = \infty$.
Case 2 : $r_\beta \in C$. There are three subcases to investigate.
Case 2.1 : $n = p$ such that p is a prime number. Since V_0 is a cyclic subgroup of $\mathbb{Z}_n$ and n is prime, we have $V_0 = \{0, 1, ..., n-1\} = \mathbb{Z}_n$.
Case 2.1.1 : $\theta \neq 1$.
By Theorem 1.6, we get $d((c, l_i, r_\eta), b) = d((c, l_1, r_\eta), (f, l_\theta, r_\beta)) = \infty$.
Case 2.1.2 : $\theta = 1$.
Case 2.1.2.1 : $f = c$. Choose $(a, l_1, r_\beta), ((k-1)a, l_1, r_\beta) \in A$. Consider

$$\begin{aligned} (c, l_1, r_\eta)((k-1)a, l_1, r_\beta) &= (c + (k-1)a, l_1, r_\beta) \text{ and then} \\ (c + (k-1)a, l_1, r_\beta)(a, l_1, r_\beta) &= (c + ka, l_1, r_\beta) \\ &= (c, l_1, r_\beta) \\ &= (f, l_\theta, r_\beta). \end{aligned}$$

It follows that $d((c, l_1, r_\eta), b) = 2$.
Case 2.1.2.2 : $f \neq c$.
Since $(f - c, l_1, r_\beta) \in A$, we obtain

$$b = (f, l_1, r_\beta) = (c, l_1, r_\eta)(f - c, l_1, r_\beta).$$

This implies that $d((c, l_1, r_\eta), b) = 1$. Hence we can conclude that

$$d((c, l_1, r_\eta), b) = 1 \text{ or } d((c, l_1, r_\eta), b) = 2 \text{ or } d((c, l_1, r_\eta), b) = \infty.$$

Case 2.2 : $n \geq 4$ such that $V_0 = \mathbb{Z}_n$.
The proof of this case is similar to the proof of Case 2.1.
Case 2.3 : $n \geq 4$ such that $V_0 \subsetneq \mathbb{Z}_n$.
Let $V_0 + 1$ denote the set $\{1, a+1, ..., (k-1)a+1\}$.
Case 2.3.1 : $n \geq 4$ such that $\dfrac{|\mathbb{Z}_n|}{|V_0|} = 2$. Then V_0 and $V_0 + 1$ are only distinct cosets of V_0 in $\mathbb{Z}_n$.
Case 2.3.1.1 : $\theta \neq 1$.
By Theorem 1.6, we get $d((c, l_1, r_\eta), b) = d((c, l_1, r_\eta), (f, l_\theta, r_\beta)) = \infty$.
Case 2.3.1.2 : $\theta = 1$.
Case 2.3.1.2.1 : $f = c$.
The proof of this case is similar to the proof of Case $f = c$ which was considered in Case 2.1.2.1.
Case 2.3.1.2.2 : $f \neq c$.
Case 2.3.1.2.2.1 : $f \neq 0$ and $c = 0$.
Case 2.3.1.2.2.1.1 : $f \neq 0$ and $f \in V_0$. Then $b = (f, l_1, r_\beta) = (za, l_1, r_\beta)$ and thus $b = (f, l_1, r_\beta) = (za, l_1, r_\beta) = (0, l_1, r_\eta)(za, l_1, r_\beta)$.
Hence $d((c, l_1, r_\eta), (f, l_1, r_\beta)) = d((0, l_1, r_\eta), b) = 1$.
Case 2.3.1.2.2.1.2 : $f \neq 0$ and $f \in \mathbb{Z}_n \setminus V_0$.
Suppose that $d((c, l_1, r_\eta), b) = d$ where $1 \leq d < \infty$. We obtain

$$\begin{aligned}
(c, l_1, r_\eta)(s_1 a, l_1, r_\beta) &= (0, l_1, r_\eta)(s_1 a, l_1, r_\beta) \\
&= (s_1 a, l_1, r_\beta), \\
(s_1 a, l_1, r_\beta)(s_2 a, l_1, r_\beta) &= (s_1 a + s_2 a, l_1, r_\beta), \\
&\vdots \\
\left(\sum_{m=1}^{d-1} s_m a, l_1, r_\beta\right)(s_d a, l_1, r_\beta) &= \left(\left(\sum_{m=1}^{d} s_m\right) a, l_1, r_\beta\right) \\
&= b
\end{aligned}$$

for some $(s_1 a, l_1, r_\beta), (s_2 a, l_1, r_\beta), ..., (s_d a, l_1, r_\beta) \in A$. Thus $f = \sum_{m=1}^{d} s_m a \in V_0$ which contradicts the fact that $f \in \mathbb{Z}_n \setminus V_0$. Then $d((c, l_1, r_\eta), b) = \infty$.
Case 2.3.1.2.2.2 : $f \neq 0$ and $c \neq 0$.

Case 2.3.1.2.2.2.1 : $f, c \in V_0$. Then $f = ua$ and $c = va$ where $1 \leq u < v \leq k-1$. Consider

$$b = (f, l_1, r_\beta) = (ua, l_1, r_\beta) = (va, l_1, r_\eta)((u + (-v))a, l_1, r_\beta)$$

for some $((u + (-v))a, l_1, r_\beta) \in A$, we obtain $d((c, l_1, r_\eta), b) = 1$.
Case 2.3.1.2.2.2.2 : $f, c \in V_1$. Then $f = ua + 1$ and $c = va + 1$ where $1 \leq u < v \leq k - 1$. Consider

$$b = (f, l_1, r_\beta) = (ua + 1, l_1, r_\beta) = (va + 1, l_1, r_\eta)((u + (-v))a, l_1, r_\beta)$$

for some $((u + (-v))a, l_1, r_\beta) \in A$, we have $d((c, l_1, r_\eta), b) = 1$.
Case 2.3.1.2.2.2.3 : $f \in V_0$ and $c \in V_1$.
Then $c = va + 1$ where $1 \leq v \leq k - 1$.
Suppose that $d((c, l_1, r_\eta), b) = d$ where $1 \leq d < \infty$. We observe that

$$\begin{aligned}(va + 1, l_1, r_\eta)(s_1 a, l_1, r_\beta) &= (va + s_1 a + 1, l_1, r_\beta),\\ &\vdots\\ (va + \sum_{m=1}^{d-1} s_m a + 1, l_1, r_\beta)(s_d a, l_1, r_\beta) &= ((va + \sum_{m=1}^{d} s_m a + 1, l_1, r_\beta)\\ &= b\end{aligned}$$

for some $(s_1 a, l_1, r_\beta), (s_2 a, l_1, r_\beta), ..., (s_d a, l_1, r_\beta) \in A$. Therefore,

$$f = va + \sum_{m=1}^{d} s_m a + 1 \in V_0$$

which contradicts the fact that $V_0 \cap V_1 = \emptyset$. Then $d((c, l_1, r_\eta), b) = \infty$.
Case 2.3.1.2.2.2.4 : $f \in V_1$ and $c \in V_0$. The proof of this case is similar to the proof of Case 2.3.1.2.2.2.3. It follows that $d((c, l_1, r_\eta), b) = \infty$.
Case 2.3.2 : $n \geq 4$ such that $\frac{|\mathbb{Z}_n|}{|V_0|} \geq 3$. Then there exist at least 3 cosets of V_0 in $\mathbb{Z}_n$. We now let $x, y \in \mathbb{Z}_n \setminus \{0\}$. Moreover, let

$V_x = \{x, a + x, ..., (k - 1)a + x\}$ and
$V_y = \{y, a + y, ..., (k - 1)a + y\}$ be distinct cosets in $\mathbb{Z}_n$.

Case 2.3.2.1 : $\theta \neq 1$.
By Theorem 1.6, we get $d((c, l_1, r_\eta), b) = d((c, l_1, r_\eta), (f, l_\theta, r_\beta)) = \infty$.

Case 2.3.2.2 : $\theta = 1$. The following possibilities have to be considered.
Case 2.3.2.2.1 : $f, c \in V_0$. The proof is similar to the proof of Case 2.3.1.2.2.2.1.
Case 2.3.2.2.2 : $f, c \in V_x$. The proof is similar to the proof of Case 2.3.1.2.2.2.2.
Case 2.3.2.2.3 : $f, c \in V_y$. The proof is similar to the proof of Case 2.3.1.2.2.2.2.
The remaining cases are provided that f and c belong to distinct cosets. Furthermore, the proofs are similar to the proof of Case 2.3.1.2.2.2.3. Thus $d((c, l_1, r_\eta), b) = \infty$. By Theorem 1.6, we obtain $d((c, l_i, r_\eta), b) = 1$ or $d((c, l_i, r_\eta), b) = 2$ or $d((c, l_i, r_\eta), b) = \infty$ for all $i \in \{1, 2..., l\}$. □

Next, we present the distance between two vertices of the Cayley digraph Γ of a rectangular group such that the first projection of a connection set of Γ is a cyclic subgroup of $\mathbb{Z}_n$.

Theorem 2.2. *Let n be a positive integer greater than or equal 2. Let Γ be the Cayley digraph of a rectangular group $\mathbb{Z}_n \times L \times R$ with respect to the connection set $A = V_0 \times B \times C$ such that $V_0 = \langle a \rangle$ is a cyclic subgroup of $\mathbb{Z}_n, B \subseteq L$ and $C \subseteq R$. For each $(c, l_i, r_\eta), (f, l_\theta, r_\beta) \in V(\Gamma)$,*

$$d((c, l_i, r_\eta), (f, l_\theta, r_\beta)) = \begin{cases} 1 & \text{if } f, c \in V_x, i = \theta \text{ and } r_\beta \in C; \\ \infty & \text{if otherwise.} \end{cases}$$

for some $x \in \{0, 1, 2, \ldots, a-1\}$.

Proof. The proof is similar to the proof of Theorem 2.1. □

3 Conclusion

In this paper, we have presented the characterizations of vertex distances $d(u, v)$ for two distinct elements $u, v \in V(\Gamma)$ where Γ is the Cayley digraph of a rectangular group with respect to the connection set A. Indeed, the first projection of a connection set considered in this paper has been separated into two types; that is, $p_1(A)$ is a cyclic subgroup of $\mathbb{Z}_n$ and $p_1(A)$ is a cyclic subgroup of $\mathbb{Z}_n$ except for the zero element.

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References

[1] J. Boland, F. Buckley, M. Miller, Eccentric Digraphs, Discrete Math., **286**, nos. 1–2, (2004), 25–29.

[2] C. Delorme, Cayley Digraphs and Graphs, European J. Combin, **34**, no. 8, (2013), 1307–1315.

[3] J. M. Howie, Fundamentals of Semigroup Theory, Oxford University Press, London, 1995.

[4] B. Khosravi, M. Mahmoudi, On Cayley Graphs of Rectangular Groups, Discrete Math., **310**, no. 4, (2010), 804–811.

[5] J. Meksawang, S. Panma, Isomorphism Conditions for Cayley Graphs of Rectangular Groups, Bull. Malays. Math. Sci. Soc., **39**, (2016), 29–41.

[6] N. Nupo, S. Panma, Independent Domination Number in Cayley Digraphs of Rectangular Groups, Discrete Math. Algorithms Appl., **10**, no. 2, (2018), https://doi.org/10.1142/S1793830918500246

[7] N. Nupo, S. Panma, Certain Structural Properties for Cayley Regularity Graphs of Semigroups and their Theoretical Applications, AIMS math., **8**, no. 7, (2023), 16228–16239.

[8] S. Panma, Characterization of Cayley Graphs of Rectangular Groups, Thai J. Math., **8**, no. 3, (2010), 535–543.

[9] S. Panma, U. Knauer, S. Arworn, On Transitive Cayley Graphs of Strong Semilattices of Right (Left) Groups, Discrete Math., **309**, no. 17, (2009), 5393–5403.

[10] S. Panma, N. Nupo, On the Independence Number of Cayley Digraphs of Rectangular Groups, Graphs Combin., **34**, (2018), 579–598.

[11] R. J. Wilson, Introduction to Graph Theory, Addison Wesley Longman Limited, England, 1996.