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On the Diophantine equation $675^x + 896^y = z^2$

Nongluk Viriyapong[1], **Jenjira Puiwong**[2]

[1]Mathematics and Applied Mathematics Research Unit
Department of Mathematics
Mahasarakham University
Maha Sarakham, 44150, Thailand

[2]Faculty of Education
Rajabhat Mahasarakham University
Maha Sarakham, 44000, Thailand

email: nongluk.h@msu.ac.th, jenjira.pu@rmu.ac.th

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Abstract

In this article, we establish that there is a unique non-negative solution to the Diophantine equation $675^x + 896^y = z^2$. The solution is $(x, y, z) = (1, 0, 26)$.

1 Introduction

Numerous mathematical studies have focused on the Diophantine equations of the type $a^x + b^y = z^2$, where a and b are fixed. For instance, in 2014, Sroysang [3] demonstrated that the Diophantine equation $143^x + 145^y = z^2$ has a unique non-negative integer solution (x, y, z), which is $(1, 0, 12)$. In 2023, N. Viriyapong and C. Viriyapong [5] proved that there are only two non-negative integer solutions (x, y, z) to the Diophantine equation $255^x + 323^y = z^2$, which are $(1, 0, 6)$ and $(0, 1, 18)$. Then, in 2024, N. Viriyapong and

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C. Viriyapong [6] showed that the Diophantine equation $147^x + 741^y = z^2$ has no non-negative integer solution.

In this paper, we focus our attention on the Diophantine equation $675^x + 896^y = z^2$, in which x, y, and z are non-negative integers.

2 Preliminaries

Throughout this paper, $a \equiv_m b$ is used to indicate that a is congruent to b modulo m, where a, b, and m are integers such that $m \geqslant 1$. To further say $a \equiv_m b$ or $a \equiv_m c$, we will write $a \equiv_m b, c$.

Now, we recall the Catalan's conjecture [1] dating back to 1844, was proved by Mihailescu [2] in 2004.

Theorem 2.1 (Catalan's conjecture). *The Diophantine equation* $a^x - b^y = 1$ *has a unique solution* $(a, b, x, y) = (3, 2, 2, 3)$, *where* a, b, x *and* y *are integers with* $\min\{a, b, x, y\} > 1$.

Now, we mention two lemmas that follow from the Catalan's conjecture.

Lemma 2.2. $(1, 26)$ *is the unique non-negative integer solution* (x, z) *for the Diophantine equation* $675^x + 1 = z^2$.

Proof. Assume that there exist non-negative integers x and z such that $675^x + 1 = z^2$. It is clear that $x \geqslant 1$. By Theorem 2.1, $x = 1$. This implies that $z = 26$. This completes the proof. □

Lemma 2.3. *The Diophantine equation* $1 + 896^y = z^2$ *has no solutions in non-negative integers.*

Proof. Assume that there exist non-negative integers y and z such that $1 + 896^y = z^2$. Obviously, $y \geqslant 1$. By Theorem 2.1, $y = 1$. This implies that $z^2 = 896$ which contradicts z being an integer. The proof is complete. □

Finally, we recall the following lemma [4].

Lemma 2.4. [4] *If* z *is an integer, then* $z^2 \equiv_{13} 0, 1, 3, 4, 9, 10, 12$.

3 Main Results

Now, we will present our main result.

Theorem 3.1. *The Diophantine equation* $675^x + 896^y = z^2$ *has the unique non-negative integer solution* $(x, y, z) = (1, 0, 26)$.

Proof. Assume that there exist non-negative integers x, y, and z such that $675^x + 896^y = z^2$. By Lemma 2.2 and 2.3, it follows that $(x, y, z) = (1, 0, 26)$ is a solution of this equation. Now, we consider $x \geqslant 1$ and $y \geqslant 1$. If y is odd, $z^2 = 675^x + 896^y \equiv_3 2$, which is the contrary to fact that $z^2 \equiv_3 0, 1$. Then y is even. Thus, $z^2 = 675^x + 896^y \equiv_4 3$ if x is odd, which contradicts the fact that $z^2 \equiv_4 0, 1$. Consequently, x is even. Since $675 \equiv_{13} 12$ and $896 \equiv_{13} 12$, we have $z^2 \equiv_{13} 2$, which contradicts Lemma 2.4. The proof is now complete. ☐

Corollary 3.2. *The Diophantine equation* $678^x + 896^y = z^4$ *has no solution in non-negative integers.*

4 Conclusion

We have proved that $(1, 0, 26)$ is the unique solution for the Diophantine equation $675^x + 896^y = z^2$, where x, y, and z are non-negative integers.

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