International Journal of Mathematics and Computer Science
Volume **20,** Issue no. 1, (2025), 439–446
DOI: https://doi.org/10.69793/ijmcs/01.2025/munshid

Chained Hesitant Fuzzy $\mathcal{R}$-Modules

Khaled Samiea Munshid[1], **Haneen Hameed Muttair**[1], **Saba Salah Majeed**[2]

[1]College of Computer Science and Information Technology
University of Sumer
Thi Qar, Iraq

[2]Department of Petroleum Engineering
College of Engineering
University of Baghdad
Baghdad, Iraq

email: khaled.samiea@uos.edu.iq, haneen.hameed@uos.edu.iq, saba.s@coeng.uobaghdad.edu.iq

(Received November 7, 2024, Accepted December 9, 2024, Published December 15, 2024)

Abstract

Suppose $\mathcal{R}$ is a commutative ring with unity. The main goal of this study is to expand on the idea of chained hesitant fuzzy $\mathcal{R}$-modules by building upon the concept of chained fuzzy $\mathcal{R}$-modules. By introducing this extension, we provide a more comprehensive framework for understanding the behavior of fuzzy modules over commutative rings. Moreover, we introduce hesitant fuzzy-level concept subsets within this new framework. This concept offers a unique perspective on the structure of fuzzy $\mathcal{R}$-modules, allowing for a more nuanced analysis of their properties and interactions within the context of commutative rings. Overall, our exploration of chained hesitant fuzzy $\mathcal{R}$-modules and hesitant fuzzy level subsets enrich the existing literature on fuzzy modules and contribute to the ongoing discussions within the field of algebra and commutative ring theory. First, some examples and

Key words and phrases: Chained fuzzy $\mathcal{R}$-modules, Hesitant fuzzy $\mathcal{R}$-modules, Chained hesitant fuzzy $\mathcal{R}$-modules, Hesitant fuzzy level subset.
AMS Subject Classifications:16D50.
ISSN 1814-0432, 2025, https://future-in-tech.net

properties of these concepts were provided. Uniform hesitant fuzzy $\mathcal{R}$-module is a concept that delve into, showcasing its defining characteristics and applications. Furthermore, the connection between chained hesitant fuzzy $\mathcal{R}$-modules and uniform hesitant fuzzy $\mathcal{R}$-modules will be investigated.

1 Introduction

This study introduces the chained hesitant fuzzy concept $\mathcal{R}$-modules as an expansion of the idea of chained fuzzy $\mathcal{R}$-modules and the concept of the hesitant fuzzy level subset in ordinary algebra. There are three main parts in this paper. In the First part, we define some key terms that will be used throughout the following sections. It is crucial to have a clear understanding of these definitions to avoid any confusion or misinterpretation of the results presented. Next, we discuss some important results that serve as the foundation for the upcoming analysis. These results have been proven and widely accepted. Moving on to the second part, some findings regarding the relationship between chained hesitant fuzzy $\mathcal{R}$-modules and their corresponding levels. In the third part of this paper, we provide some applications on the main topic.

2 Preliminaries

To proceed further, it is important to establish some key definitions and results that will be essential for reference. This foundational knowledge provides a solid framework for the concepts that will be explored in more detail later on.

Definition 2.1. *[1] Assume that X is a non-empty set. The set A of hesitant fuzzy on X is defined as a function (x) When implemented on $H_A(x)$, gives back a limited subset of $[0,1]$; i.e., $A = \{(x, H_A(x))\,|x{\in}X\}$, For convenience, $H_A(x)$ is a collection of some difference values in [0,1] that indicate the potential membership of the element $x{\in}X$ to A.*

Definition 2.2. *[1] Suppose M is an $\mathcal{R}$-module. A subset X of M is called a fuzzy module of an $\mathcal{R}$-module M if X satisfies the following conditions:*
(1) $X(\alpha - \beta) \geq min\{X(\alpha), X(\beta)\}, \forall\alpha, \beta \in M$.
(2) $X\ (r\alpha) \geq X(\alpha)$, for $\forall\ \alpha \in\ M$ and $r \in\ \mathcal{R}$.
(3) $X\ (0) = 1$, (0 represents the zero element of M).

Definition 2.3. *[2] Let X and Y be two fuzzy modules of an $\mathcal{R}$-module M. Y is called a fuzzy submodule of X if $Y \subseteq X$.*

Definition 2.4. *[2] Assume H is a set of hesitant fuzzy over $\mathcal{R}$-module M. Then H is called a hesitant fuzzy module (for short HFM) over M if $\forall\alpha, \beta \in M$, $r \in \mathcal{R}$.*
(1) $H(\alpha - \beta) \supseteq H(\alpha) \bigcap H(\beta)$.
(2) $H(r\alpha) \supseteq H(\alpha)$.

Definition 2.5. *[3] An $\mathcal{R}$-module M is called a chained module if, for submodules X and Y of M, either $X \subseteq Y$ or $Y \subseteq X$.*

Definition 2.6. *[3] Suppose H is a fuzzy module of an $\mathcal{R}$-module M. We say that H is a chained fuzzy $\mathcal{R}$-module if, for every fuzzy-submodule of H, either $X \subseteq Y$ or $Y \subseteq X$.*

Definition 2.7. *[2] Let H be a reference set and let X and Y be hesitant fuzzy subsets of H. We write $X \subseteq Y$ if $X(\alpha) \subseteq Y(\alpha)$ for all $\alpha \in H$.*

Definition 2.8. *[4],[5] Let X and Y be fuzzy sets in H. Then*
(1) $X = Y$ if and only if $X\ (\alpha) = Y(\alpha)$, for all $\alpha \in H$.
(2) $X \subseteq Y$ if and only if $X\ (\alpha) \leq Y(\alpha)$, for all $\alpha \in H$.
(3) $X \cap Y(\alpha) = \min\{X(\alpha), Y(\alpha)\}$ for all $\alpha \in H$.
(4) $X \cup Y(\alpha) = \max\{X(\alpha), Y(\alpha)\}$ for all $\alpha \in H$

Definition 2.9. *[3] Let X be a fuzzy subset of A. For all $t \in [0, 1]$, the set $X_t = \{\alpha \in A,\ X(\alpha) \geq t\}$ is called a level subset of X.*

Lemma 2.10. *[3],[6]*
1- Assume that M_1 and M_2 are two modules on $\mathcal{R}$. Let $f : M_1 \to M_2$ be an epimorphism and suppose that C_1 is fuzzy submodule from M_1. Then $f(C_1)$ is a fuzzy submodule of M_2.
2- Let M_1 and M_2 be two $\mathcal{R}$-modules. Let $f : M_1 \to M_2$ is a homomorphism and suppose that C_2 is a fuzzy-submodule of M_2. Hence $f^{-1}(\ C_2)$ is a fuzzy submodule of M_1.

3 Chained Hesitant Fuzzy $\mathcal{R}$-Modules

Definition 3.1. *Assume that C is a hesitant fuzzy $\mathcal{R}$-module of a module M. Then C is a chained hesitant fuzzy $\mathcal{R}$-module if, for each hesitant fuzzy submodules X and Y of C either $X \subseteq Y$ or $Y \subseteq X$.*

Proposition 3.2. *Let C_1 and C_2 be two chained hesitant fuzzy $\mathcal{R}$-modules of an $\mathcal{R}$-module M. Then $C_1 \cup C_2$ is a chained hesitant fuzzy $\mathcal{R}$-module of M if $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$.*

Proof. Suppose that X and Y are hesitant fuzzy-submodules of an $\mathcal{R}$-module $C_1 \cup C_2$. Since $C_1 \subseteq C_2$, then $C_1 \cup C_2 = C_2$. Then X and Y are two hesitant fuzzy submodules of C_2. Since C_2 is a chained hesitant fuzzy $\mathcal{R}$-module from an $\mathcal{R}$-module M, $X \subseteq Y$ or $Y \subseteq X$. As a result, $C_1 \cup C_2$ is a chained hesitant fuzzy $\mathcal{R}$-module of $\mathcal{R}$-module M.
If $C_2 \subseteq C_1$, $C_1 \cup C_2 = C_1$. Since C_1 is a chained hesitant fuzzy $\mathcal{R}$-module of $\mathcal{R}$-module M, $C_1 \cup C_2$ is a chained hesitant fuzzy $\mathcal{R}$-module of $\mathcal{R}$-module M. □

Proposition 3.3. *Suppose C_1 and C_2 are two chained hesitant fuzzy $\mathcal{R}$-modules of an $\mathcal{R}$-module M. Then $C_1 \cap C_2$ is a chained hesitant fuzzy $\mathcal{R}$-module.*

Proof. Let X and Y be hesitant fuzzy-submodules of an $\mathcal{R}$-module M. Since $C_1 \cap C_2 \subseteq C_1$ and $C_1 \cap C_2 \subseteq C_2$, X and Y represent hesitant fuzzy-submodules of C_1 and hesitant fuzzy-submodules of C_2. Since C_1 and C_2 are chained hesitant fuzzy $\mathcal{R}$-modules from $\mathcal{R}$-module M, $X \subseteq Y$ or $Y \subseteq X$. Thus, $C_1 \cap C_2$ is a chained hesitant fuzzy $\mathcal{R}$-module. □

Theorem 3.4. *Let $\{C_i / i \in I\}$ be a family of chained hesitant fuzzy $\mathcal{R}$-modules of $\mathcal{R}$-module M. Then $\bigcap_{i \in I} C_i$ is a chained hesitant fuzzy $\mathcal{R}$-module.*

Proof. Suppose X and Y represent two hesitant fuzzy-submodules of $\bigcap_{i \in I} C_i$. Then X, Y are hesitant fuzzy-submodules of $C_i, \forall i \in I$. Since C_i chained hesitant fuzzy $\mathcal{R}$-module $\forall i \in I$, $X \subseteq Y$ or $X \subseteq Y$. Thus, $\bigcap_{i \in I} C_i$ is a hesitant fuzzy chained $\mathcal{R}$-module. □

Theorem 3.5. *Let $\{C_i / i \in I\}$ be a family of chained hesitant fuzzy $\mathcal{R}$-modules of $\mathcal{R}$-module M. Then $\cup_{i \in I} C_i$ is a chained hesitant fuzzy $\mathcal{R}$-module if $C_i \subseteq C_j$ or $C_j \subseteq C_i$ $\forall i, j \in I$.*

Proof. Assume X and Y are two hesitant fuzzy-sub-modules of $\cup_{i \in I} C_i$. There exists $k \in I$ and X, Y are two hesitant fuzzy-submodules of C_k. Since C_k is a chained hesitant fuzzy $\mathcal{R}$-module, $X \subseteq Y$ or $X \subseteq Y$. □

Theorem 3.6. *A module of a hesitant fuzzy C of $\mathcal{R}$-module M is a chained hesitant fuzzy $\mathcal{R}$-module if and only if C_λ is a chained $\mathcal{R}$-module, $\forall \lambda \in (0, 1]$.*

Proof. Suppose that C is a chained hesitant fuzzy $\mathcal{R}$-module. We prove that C_λ is a chained $\mathcal{R}$-module, $\forall\lambda\in(0,1]$. For submodules I and J of C_k, define
$X(\alpha)=\begin{cases}\{\lambda\} & \alpha\in I\\ \{0\} & \alpha\notin I\end{cases}, \quad Y(\alpha)=\begin{cases}\{\lambda\} & \alpha\in J\\ \{0\} & \alpha\notin J\end{cases}$
where X and Y are hesitant fuzzy submodules of C. However, $X_\lambda = I, Y_\lambda = J$. Since C is a chained hesitant fuzzy $\mathcal{R}$-module, either $X \subseteq Y$ or $Y \subseteq X$. Hence, $X_\lambda\subseteq Y_\lambda$ or $Y_\lambda\subseteq X_\lambda$ by Lemma 1.8 in [3]. Thus, $I\subseteq J$ or $J\subseteq I$.
Conversely, suppose that C_λ is chained $\mathcal{R}$-module. To prove that C is a hesitant fuzzy $\mathcal{R}$-module, let X and Y be two hesitant fuzzy sub-modules of C. Then, X_λ, Y_λ are submodules of C_λ, for every $\lambda\in(0,1]$. Since C_λ is a chained $\mathcal{R}$-module, $X_\lambda\subseteq Y_\lambda$ or $Y_\lambda\subseteq X_\lambda$. Hence, $X \subseteq Y$ or $Y \subseteq X$ by Lemma 1.8 in [3]. □

Proposition 3.7. *If $C_1\subseteq C_2$ and C_2 is a chained hesitant fuzzy $\mathcal{R}$-module, then C_1 is a chained hesitant fuzzy $\mathcal{R}$-module.*

Proof. Let X and Y be two hesitant fuzzy submodules of C_1. Since $C_1\subseteq C_2$, X and Y are two hesitant fuzzy-submodules of C_2. Since C_2 is a chained hesitant fuzzy $\mathcal{R}$-module, $X \subseteq Y$ or $Y \subseteq X$ which suggests that C_1 is a chained hesitant fuzzy $\mathcal{R}$-module. □

Definition 3.8. *A hesitant fuzzy $\mathcal{R}$-module C is called a uniform hesitant fuzzy $\mathcal{R}$-module if $C_1\cap C_2\neq\{0_1\}$ for each nontrivial hesitant fuzzy-submodules C_1 and C_2 of C.*

Proposition 3.9. *A hesitant fuzzy module C from an $\mathcal{R}$-module M is a uniform hesitant fuzzy $\mathcal{R}$-module if and only if C_λ is a uniform fuzzy $\mathcal{R}$-module, $\forall\lambda\in(0,1]$.*

Proof. Let C be a uniform fuzzy $\mathcal{R}$-module. We prove that C_λ is a uniform fuzzy $\mathcal{R}$-module $\forall\lambda\in(0,1]$. For submodules I and J of C_λ, define
$X(\alpha)=\begin{cases}\{\lambda\} & \alpha\in I\\ \{0\} & \alpha\notin I\end{cases}, \quad Y(\alpha)=\begin{cases}\{\lambda\} & \alpha\in J\\ \{0\} & \alpha\notin J\end{cases}$
where X and Y are two hesitant fuzzy submodules of C. But $X_\lambda = I, Y_\lambda = J$. Since C is a uniform hesitant fuzzy $\mathcal{R}$-module, $(X\cap Y)_\lambda\neq\{0_1\}$ which implies that $X_\lambda\cap Y_\lambda\neq\{0_1\}$ using Remark 1.5 in [3].
Conversely, suppose that C_λ is a uniform fuzzy $\mathcal{R}$-module. To prove that C is uniform hesitant fuzzy $\mathcal{R}$-module, let X and Y be two hesitant fuzzy-submodules of C. Then X_λ, Y_λ are fuzzy-submodules in C_λ, for all $\lambda\in(0,1]$. Since C_t is a uniform fuzzy $\mathcal{R}$-module, $X_\lambda\cap Y_\lambda\neq\{0_1\}$, which implies that $(X\cap Y)_\lambda\neq\{0_1\}$ using Remark 1.5 in [3]. Thus $X\cap Y\neq\{0_1\}$. □

Now, we present a proposition about the relationship between the uniform hesitant fuzzy $\mathcal{R}$-module and the chained hesitant fuzzy $\mathcal{R}$-module.

Proposition 3.10. *Every chained hesitant fuzzy $\mathcal{R}$-module is a uniform hesitant fuzzy $\mathcal{R}$-module.*

Proof. Let C be a chained hesitant fuzzy $\mathcal{R}$-module of an $\mathcal{R}$-module M and let X and Y be two hesitant fuzzy submodules of C. Since C is a chained hesitant fuzzy $\mathcal{R}$-module, $X \subseteq Y$ or $Y \subseteq X$.
If $X \subseteq Y$, then $X \cap Y = X$ and if $Y \subseteq X$, then $Y \cap X = Y$.
This implies that $X \cap Y \neq \{0_1\}$. □

The following example illustrates the converse of Proposition 3.9 is not true.

Example 3.11. *Consider $M = Z$ as a Z-module. Let $C : M \to [0,1]$ be such that $C(\alpha) = 1$, $\forall \alpha \in M$. $C_\lambda = z$ for all $\lambda \in [0,1]$. But Z is a uniform fuzzy Z-module. Hence. by Proposition 3.10, C is a uniform hesitant fuzzy Z-module. But C is not a chained hesitant fuzzy Z-module since there exist hesitant fuzzy-submodules X and Y of C, defined by:*

$$X(\alpha) = \begin{cases} \{1\} & \alpha \in (2) \\ \{\frac{1}{4}\} & \alpha \notin (2) \end{cases}, \quad Y(\alpha) = \begin{cases} \{1\} & \alpha \in (5) \\ \{\frac{1}{4}\} & \alpha \notin (5) \end{cases}$$

and $X \nsubseteq Y$ and $Y \nsubseteq X$.

4 Applications of Chained Hesitant Fuzzy $\mathcal{R}$-Modules

1. **Decision Support Systems (DSS):** Chained Hesitant Fuzzy $\mathcal{R}$-modules may be used in decision support systems to handle intricate decision-making situations. When dealing with decision-making in DSS, which involves various criteria and uncertainties, the use of hesitant fuzzy sets and $\mathcal{R}$-modules may provide a more thorough framework for modelling and analyzing choice variables [7].

2. **Control Systems:** Chained Hesitant Fuzzy $\mathcal{R}$-modules may be used in control systems, particularly in the management of imprecise parameters and unpredictable situations, such as those seen in industrial processes or autonomous systems. This improves the flexibility and resilience of control algorithms in situations that undergo frequent changes [8].

3. **Artificial Intelligence (AI) and Machine Learning:** Integrating reluctant fuzzy sets and $\mathcal{R}$-modules into AI and machine learning models might contribute in managing fuzzy or uncertain data. This improves the capacity of these models to adjust to real-world situations with different degrees of inaccuracy in input data [9].

4. **Supply Chain Management:** Chained Hesitant Fuzzy $\mathcal{R}$-modules may have applications in optimizing supply chain procedures. This approach has the potential to enhance the resilience and efficiency of supply chain operations by tackling uncertainties and swings in demand forecasting, inventory management, and decision-making [10].

5 Conclusions

The definitions of the chained and uniform hesitant fuzzy $\mathcal{R}$-module and hesitant fuzzy level subset provided a comprehensive understanding of these concepts and their applications in various fields. Studying the relationships among different types of hesitant fuzzy submodules allows for a deeper analysis of their properties and interactions. For example, the chained hesitant fuzzy $\mathcal{R}$-module enables the process of union and intersection between any two hesitant fuzzy submodules, while the uniform hesitant fuzzy $\mathcal{R}$-module presents a fundamental concept with practical implications. The relationship among chained hesitant fuzzy $\mathcal{R}$-module and uniform hesitant fuzzy $\mathcal{R}$-module further highlights the importance of understanding these concepts in relation to one another. By exploring the similarities and differences between these two types of hesitant fuzzy modules, researchers can gain insights into their respective strengths and weaknesses. In conclusion, the study of hesitant fuzzy sub-modules offers a rich field of research with potential applications in various disciplines.

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