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About a property of first-order autonomous differential equations

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Abstract

We find the solution of a first-order autonomous differential equation defined on the entire real line. This problem is solved using elements of mathematical analysis and topology.

1 Introduction

Autonomous differential equations play a key role in modeling systems in various fields. In ecology, they capture population dynamics [1]; in chemical kinetics, they describe concentration-based reactions [2]; in economics, they model self-regulating markets [3]; in engineering, they characterize feedback and oscillatory systems [4]. Their time-invariant nature facilitates stability analysis and long-term prediction, making them essential for understanding equilibrium and cyclic behavior in complex systems.

In this paper, we investigate an autonomous equation and the process of finding solutions defined on the entire real line, taking into account imposed conditions on the solutions of the considered equation and the function on the right-hand side.

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2 Main results

Theorem 2.1. *Let $x = \varphi(t)$ be a solution of the differential equation*

$$\dot{x}(t) = f(x), \tag{2.1}$$

where $f(x)$ is a continuous function on $\mathbb{R}$ and satisfies:

1. $\varphi(t) \in C^1(\mathbb{R}\setminus\{t_0\})$,
2. $\lim_{t\to t_0} \varphi(t) = \infty$,
3. $\lim_{t\to\pm\infty} \varphi(t) = \alpha$, *and*
4. $\varphi(t)$ *is monotonically decreasing on* $\mathbb{R}$.

Then there exists a solution defined on $\mathbb{R}$.

Proof. From conditions 2 and 3, it follows that $\lim_{t\to t_0^+} \varphi(t) = \infty$ and $\lim_{t\to t_0^-} \varphi(t) = -\infty$.

Taking into account the monotonicity, the mapping φ is bijective; in particular $\varphi : (-\infty, t_0) \to (-\infty, \alpha)$ and $\varphi : (t_0, \infty) \to (\alpha, \infty)$.

Since the mapping is bijective, the function $\varphi(t)$ has an inverse $t = \varphi^{-1}(x)$, which is a continuously differentiable function on $\mathbb{R}\setminus\{\alpha\}$, with a domain of definition $\mathbb{R}\setminus\{t_0\}$.

Substituting $x = \varphi(t)$ into the original equation

$$\dot{\varphi}(t) = f(\varphi(t)),$$

and after replacing $t = \varphi^{-1}(x)$, we obtain:

$$f(x) = \dot{\varphi}\left(\varphi^{-1}(x)\right). \tag{2.2}$$

Since the function $f(x)$ is continuous by assumption, we have

$$f(\alpha) = \lim_{x\to\alpha} f(x) = \lim_{x\to\alpha} \dot{\varphi}\left(\varphi^{-1}(x)\right) = \lim_{\beta\to\infty} \dot{\varphi}(\mp\beta) = 0. \tag{2.3}$$

Taking into account (2.3), we obtain:

$$f(x) = \begin{cases} \dot{\varphi}(\varphi^{-1}(x)), & if\ x \neq \alpha \\ 0, & if\ x = \alpha \end{cases} \tag{2.4}$$

Therefore, $x = \alpha$ is a solution of the equation (2.1) defined on $\mathbb{R}$.

Example. Let us take the function $\varphi(t) = 2025^{\frac{1}{t}} + \frac{1}{t}$. Then we obtain:

$$\lim_{t \to \pm\infty} \varphi(t) = 1,$$

$$f(\varphi(t)) = -\frac{1}{t^2}\left(2025^{\frac{1}{t}} \ln 2025 + 1\right) < 0.$$

The mapping φ works as follows:

$$\varphi : (-\infty, 0) \to (-\infty, 1)$$

and

$$\varphi : (0, \infty) \to (1, \infty).$$

Next, we find $f(x)$:

$$f(x) = \begin{cases} -\frac{1}{\{\varphi^{-1}(x)\}^2}\left(2025^{\frac{1}{\varphi^{-1}(x)}} \ln 2025 + 1\right), \ if \ x \neq 1 \\ 0, \ if \, x = 1 \end{cases}.$$

Therefore, the solution defined on the entire real line is $x = 1$.

Theorem 2.2. *Suppose all conditions of Theorem 2.1 are satisfied. Then, on any given interval (t_1, t_2), the solution of the equation (2.1), defined on the specified interval, has the form $x = \varphi(t - \tau)$, $\forall \tau : (\tau + \alpha) \notin (t_1, t_2)$.*

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