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On a Comparison of Variance Estimators of 2-Parameter Exponential Distribution using Multiple Criteria Decision Making Method

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Abstract

In this paper we consider the estimators of the variance, θ^2 of the 2-parameter exponential distribution. The variance estimators consist of the minimum mean squared error estimator and a class of shrinkage estimators given prior information. The purpose of this study is to present a theorem that compares these estimators based on the Multiple Criteria Decision Making method. The results show that $\hat{\theta}^2_{2(-1)}$ is the best estimator while $\hat{\theta}^2_{2(2)}$, $\hat{\theta}^2_{2(1)}$, $\hat{\theta}^2_{2MMSE}$ and $\hat{\theta}^2_{2(-2)}$ are lower in rank respectively when sample size is greater than ten.

1 Introduction

The problem on estimation of parameter of any distribution is an interesting one. In particular, we have the estimation of parameters of 2-parameter exponential distribution such as mean and variance. If a random variable X has a 2-exponential

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distribution with scale and location parameters as θ and γ respectively, then this distribution is formulated as

$$f(x) = \frac{1}{\theta} exp(\frac{-(x-\gamma)}{\theta}), x > \gamma, \theta > 0.$$

This distribution has mean $\mu = \gamma + \theta$ and variance θ^2. Let $x_1, x_2, ..., x_n$ be a random sample of size n from an exponential population. In 1996, Tracy [2] proposed a class of shrinkage estimators for the variance given prior information θ_0. The estimators are $\hat{\theta}^2_{2(q)} = \theta_0^2 + \beta_q((\bar{x} - x_{(1)})^2 - \theta_0^2)$ where $\beta_q = \frac{n^{2q}\Gamma(n+2q-1)}{\Gamma(n+4q-1)}$ and q is non-zero real number. In 1997, Pandey and Singh [1] obtained the minimum mean squared error (MMSE) estimator of θ^2 as $\hat{\theta}^2_{2MMSE} = \frac{n^2(\bar{x}-x_{(1)})^2}{(n+1)(n+2)}$, where $x_{(1)} = \min_i\{x_i\}$.

In this paper the estimators of variance θ^2 in a 2-parameter exponential distribution are compared and ranked in terms of mean square errors using the Multiple Criteria Decision Making (MCDM) method. The MCDM method is briefly described in section 2. Section 3 contains the mean square errors of each estimator and section 4 describes the main results.

2 Description of MCDM procedure

As advocated by Zeleny [3] and Yoon and Hwang [4], Multiple Criteria Decision making (MCDM) is a technique that can be used for an assessment and a decision making where the multiple criteria are presented. MCDM is a method to integrate the multiple risks $(x_{i1}, ..., x_{iN})$ for the i^{th} estimator into a single meaningful and overall risk factor (see [5,6,7]). In the context of a discrete risk matrix $X = (x_{ij}) : K \times N$ where $x_{ij}\prime s$ represent risk of the i^{th} estimator for the j^{th} parameter point, and we need to compare K estimators simultaneously with respect to all the N parameter points. In 2004, Lertprapai [8] proposed a continuous version of this setup which is relevant to our problem and which would involve $x_{ij}\prime s$ representing risks or mean squared errors where the index j would vary continuously. Moreover, Lertprapai [8] proved a general result in the case of L_1-norm. Suppose $M_1, M_2, ..., M_K$ are some estimators to be compared with respect to their mean squared errors $MSE(M_i) = x_i(\lambda)$, $i = 1, 2, ..., K$, where $\underline{\lambda} \leq \lambda \leq \overline{\lambda}$. Then, the i^{th} estimators are better than the j^{th} estimators if

$$\int_{\underline{\lambda}}^{\overline{\lambda}} x_i(\lambda) w(\lambda) d\lambda < \int_{\underline{\lambda}}^{\overline{\lambda}} x_j(\lambda) w(\lambda) d\lambda, \quad (2.1)$$

where $w(\lambda)$ is the weight function. Hence, we can use inequality (2.1) to compare the estimators. $\hat{\theta}^2_{2MMSE}$ and $\hat{\theta}^2_{2(q)}$ when $q = -2, -1, 1, 2$ for estimation of $\hat{\theta}^2$. Therefore, $x_i\prime s$ represent the mean squared errors of the five estimators which are functions of a real variable λ.

3 Mean squared errors

Let $\lambda = \frac{\theta_0^2}{\theta^2}$. The $MSEs$ of $\hat{\theta}^2_{2MMSE}$ and $\hat{\theta}^2_{2(q)}$ are given as follows [2]:

$$MSE(\hat{\theta}^2_{2MMSE}) = \frac{2(2n+1)}{(n+1)(n+2)}, \tag{3.1}$$

$$MSE(\hat{\theta}^2_{2(q)}) = \lambda^2(1-\beta_q)^2 - 2\lambda\left\{1 + \tfrac{n-1}{n}\beta_q^2 - \tfrac{2n-1}{n}\beta_q\right\} + 1$$

$$+\frac{(n-1)(n+1)(n+2)}{n^3}\beta_q^2 - 2\frac{(n-1)}{n}\beta_q \tag{3.2}$$

where $\beta_q = \frac{n^{2q}\Gamma(n+2q-1)}{\Gamma(n+4q-1)}$.

4 Main result

Following [8], a general result to compare the estimators using inequality (2.1) for $n \geq 10$ the ordering of the variance estimators $\hat{\theta}^2$ is shown in the following theorem.

Theorem 4.1 If the estimators of the $\hat{\theta}^2$ in a 2-parameter exponential distribution are the minimum mean squared error (MMSE) estimator ($\hat{\theta}^2_{2MMSE}$) and a class of shrinkage estimators ($\hat{\theta}^2_{2(q)}, q = -2, -1, 1, 2$), then, based on mean squared errors, $\hat{\theta}^2_{2(-1)}$ is the best estimator while $\hat{\theta}^2_{2(2)}$,$\hat{\theta}^2_{2(1)}$,$\hat{\theta}^2_{2MMSE}$ and $\hat{\theta}^2_{2(-2)}$ are lower in rank respectively for $n \geq 10$ under MCDM approach and the weight function $w(\lambda) = 1$.

Proof. Mean squared errors of $\hat{\theta}^2_{2MMSE}$ and $\hat{\theta}^2_{2(q)}, q = -2, -1, 1, 2$ are obtained from (3.1) and (3.2). Write $x_1(\lambda) = MSE(\hat{\theta}^2_{2MMSE})$, $x_2(\lambda) = MSE(\hat{\theta}^2_{2(-2)})$, $x_3(\lambda) = MSE(\hat{\theta}^2_{2(-1)})$, $x_4(\lambda) = MSE(\hat{\theta}^2_{2(1)})$, $x_5(\lambda) = MSE(\hat{\theta}^2_{2(2)})$.
From (2.1), the i^{th} estimator is better than the j^{th} estimator with $w(\lambda) = 1$. Consider $0 < \lambda < 2$. Then

$$\int_0^2 x_i(\lambda)d\lambda < \int_0^2 x_j(\lambda)d\lambda.$$

For $n \geq 10$, we can show that

$$\int_0^2 x_3(\lambda)d\lambda < \int_0^2 x_5(\lambda)d\lambda < \int_0^2 x_4(\lambda)d\lambda < \int_0^2 x_1(\lambda)d\lambda < \int_0^2 x_2(\lambda)d\lambda.$$

Comparing each pair of inequalities, we have four cases:

Case 1: $\int_0^2 x_3(\lambda)d\lambda < \int_0^2 x_5(\lambda)d\lambda$. We get

$$\int_0^2 \Bigg(\lambda^2(1-\beta_{-1})^2 - 2\lambda\Bigg\{1 + \tfrac{n-1}{n}\beta_{-1}^2 - \tfrac{2n-1}{n}\beta_{-1}\Bigg\}$$

$$+1 + \tfrac{(n-1)(n+1)(n+2)}{n^3}\beta_{-1}^2 - 2\tfrac{(n-1)}{n}\beta_{-1}\Bigg)d\lambda <$$

$$\int_0^2 \Bigg(\lambda^2(1-\beta_2)^2 - 2\lambda\Bigg\{1+\tfrac{n-1}{n}\beta_2^2 - \tfrac{2n-1}{n}\beta_2\Bigg\} + 1 + \tfrac{(n-1)(n+1)(n+2)}{n^3}\beta_2^2 - 2\tfrac{(n-1)}{n}\beta_2\Bigg)d\lambda,$$

$$\tfrac{2(12n^6-138n^5+1140n^4-4175n^3+5154n^2+960n-2400)}{3n^7} <$$

$$\tfrac{2(12n^7+321n^6+4278n^5+26473n^4+94356n^3+202644n^2+246240n+129600)}{3(n+3)^2(n+4)^2(n+5)^2(n+6)^2},$$

which we may rewrite as

$$(27n^5 - 2351688)n^8 + (1362n^3 - 680668)n^9 + (7548n - 41910)n^{10}$$

$$(5046335n^3 - 189437736)n^4 + (51774858n^3 - 696124800)n^3$$

$$+(82716996n^3 - 418003200)n^2 + (466560000n + 311040000) > 0.$$

Since each term is greater than zero for $n \geq 10$, this inequality is true.

Case 2: $\int_0^2 x_5(\lambda)d\lambda < \int_0^2 x_4(\lambda)d\lambda$. Here

$$\int_0^2 \Bigg(\lambda^2(1-\beta_2)^2 - 2\lambda\Bigg\{1 + \tfrac{n-1}{n}\beta_2^2 - \tfrac{2n-1}{n}\beta_2\Bigg\} + 1 + \tfrac{(n-1)(n+1)(n+2)}{n^3}\beta_2^2 - 2\tfrac{(n-1)}{n}\beta_2\Bigg)d\lambda <$$

$$\int_0^2 \Bigg(\lambda^2(1-\beta_1)^2 - 2\lambda\Bigg\{1 + \tfrac{n-1}{n}\beta_1^2 - \tfrac{2n-1}{n}\beta_1\Bigg\} + 1 + \tfrac{(n-1)(n+1)(n+2)}{n^3}\beta_1^2 - 2\tfrac{(n-1)}{n}\beta_1\Bigg)d\lambda,$$

$$\frac{2(12n^7+321n^6+4278n^5+26473n^4+94356n^3+202644n^2+246240n+129600)}{3(n+3)^2(n+4)^2(n+5)^2(n+6)^2} < \frac{4(6n^3+3n^2+3n+2)}{3(n+1)^2(n+2)^2},$$

which we may rewrite as

$$3n(5n^3+39n^2+114n)(3n^6+17n^5+249n^4+727n^3-960n^2-4572n-2160) > 0.$$

Since $(5n^3+39n^2+114n) > 0$ and $(3n^6+17n^5+249n^4+727n^3-960n^2-4572n-2160) > 0$ for $n \geq 10$. So this inequality is true.

Case 3: $\int_0^2 x_4(\lambda)d\lambda < \int_0^2 x_1(\lambda)d\lambda$, We obtain

$$\int_0^2 \left(\lambda^2(1-\beta_1)^2 - 2\lambda\left\{1+\tfrac{n-1}{n}\beta_1^2 - \tfrac{2n-1}{n}\beta_1\right\} + 1 + \tfrac{(n-1)(n+1)(n+2)}{n^3}\beta_1^2 - 2\tfrac{(n-1)}{n}\beta_1\right) d\lambda$$

$$< \int_0^2 \frac{2(2n+1)}{(n+1)(n+2)} d\lambda,$$

$$\frac{4(6n^3+3n^2+3n+2)}{3(n+1)^2(n+2)^2} < \frac{4(2n+1)}{(n+1)(n+2)},$$

and rewrite as $18n^2+18n+4 > 0$ or $18n(n+1)+4 > 0$ which is true for some positive integer n.

Case 4: $\int_0^2 x_1(\lambda)d\lambda < \int_0^2 x_2(\lambda)d\lambda$. In this case,

$$\int_0^2 \frac{2(2n+1)}{(n+1)(n+2)} d\lambda < \int_0^2 \left(\lambda^2(1-\beta_{-2})^2 - 2\lambda\left\{1+\tfrac{n-1}{n}\beta_{-2}^2 - \tfrac{2n-1}{n}\beta_{-2}\right\}\right.$$

$$\left. +1 + \tfrac{(n-1)(n+1)(n+2)}{n^3}\beta_{-2}^2 - 2\tfrac{(n-1)}{n}\beta_{-2}\right) d\lambda$$

which we may have

$$\begin{gathered} 207n^{11} - 531n^{10} - 76829n^9 + 1157169n^8 - 7210621n^7 \\ +18704817n^6 + 9140664n^5 - 141086856n^4 + 131724720n^3 \\ +264819024n^2 - 99719424n - 109734912 > 0. (4.1) \end{gathered}$$

By mathematical induction,we prove that this inequality is true for $n \geq 10$. Let $P(n)$ be the statement of left side of inequality (4.1). For $n = 10$, $P(10) = 536804533248 > 0$ which is true. Assume that inequality (4.1) is true for some $n > 10$. Considering $P(n+1)$, we get

$$\begin{gathered} P(n+1) = (n-9)\{(2277n^2 + 26568n - 442089)n^7 + 2000000n^6 + (500000n - 1887454)n^5 \\ +(17297n^2 - 1098393)n^4 + 135296479n^3 + 924770395n^2 + 8121764811n \\ +73572368816\} + 662328771684. \end{gathered}$$

Since each term is greater than zero when $n \geq 10$, the statement is true for $n \geq 10$.

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