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# Best Spline Approximation in Besov-Orlicz Space

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**Abstract**

Orlicz spaces have got the attention of many researchers over the decades. Many spaces have been defined in terms of Orlicz spaces, in particular Besov-Orlicz spaces. In this paper, we discuss this class of spaces with modulus of continuity. We study the best approximation of Besov-Orlicz spaces with splines and polynomials. The degree of best approximation depends on modulus of continuity.

Orlicz space is a large space of measurable functions that is important to study. Many generalizations are made to Orlicz spaces, such as, Orlicz-Lorentz [1], Quasi-Orlicz [2], Besov-Orlicz [3] and others. An Orlicz function is defined as a non-negative convex function with $\emptyset(0) = 0$, and $\lim_{t\to 0+} \frac{\phi(t)}{t} = \lim_{t\to\infty} \frac{t}{\phi(t)} = 0$. Also, define the outer function $\Psi_\phi(f) = \int_0^\alpha \phi(f^*)\, d\mu$. Many extensions were studied by several authors, such as, Orlicz-Lorentz [1], that is defined as $\{f \text{ measurable}, \Psi_\phi(\lambda f) < \infty, \text{ for some } \lambda > 0\}$, and Luxemburg norm [4] is given by

$$\|f\|_\phi = \inf_{\lambda>0} \left\{ \Psi_\phi\left(\frac{f}{\lambda}\right) \le 1 \right\}, \tag{0.1}$$

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In [3], authors define the space of Besov-Orlicz $B_{\phi,\infty}^{\omega_{\mu,v}}$ in terms of modulus of continuity, as a Banach functions subspace of $C\left[0,1\right]$ with the norm

$$\|f\|_{\phi,\infty}^{\omega_{\mu,v}} = \|f\|_{\phi} + \sup_{0<t\leq 1}\left(\frac{\omega_{\phi}\left(f,t\right)}{\omega_{\mu,v}}\right) \tag{0.2}$$

where

$$\omega_{\phi}\left(f,t\right) = \sup_{0\leq h\leq t}\left(\Delta_h'\left(f\right)\right)_{\phi}, \omega_{\mu,v}\left(t\right) = t^{\mu}\left(1+\log\left(\frac{1}{t}\right)\right)_{(x)}^{v} \tag{0.3}$$

and $\Delta_h'\left(f\left(x\right)\right) = \left[f\left(x+h\right)-f\left(x\right)\right]$, for any $x\in\left[0,1-h\right], 0<\mu<1$ and $v>0$. For any $r\in\mathbb{N}_0$, we use the symbol $\Pi_r$ to define the space of algebraic polynomials of degree less than r, and we put $\|P\| = \sup_x P(x)$, where $P\in\Pi_r$. Define the best approximation of functions $f\in B_{\phi,\infty}^{\omega_{\mu,V}}$ to the space $\Pi_r$ is the polynomial $p^*\in\Pi_r$ that satisfies $E_{r,\phi}\left(f\right) = \|f-p^*\|_{\phi} = \inf_{p\in\Pi_r}(f-p)$, where we denote $E_{r,\phi}\left(f\right)$ to be the degree of best approximation of a function $f\in B_{\phi,\infty}^{\omega_{\mu,V}}$ by a polynomial $p^*$ of degree at most $r$.

This paper deals mainly with a class of continuous piecewise polynomials, called splines [5], that are $\mathcal{S}_r\left(y_n\right)$. $\mathcal{S}_r\left(y_n\right)$ is the space of all splines of degree $r$ with knots $y_n = \left(y_i\right)_0^n, -1 = y_0 < y_1 < \ldots < y_{n-1} < y_n = 1, s\in\mathcal{S}_r\left(y_n\right)$ if $s\in\Pi_r$ on the $\left(y_i,\ y_{i+1}\right), i = 0,\cdots,n-1$. Let $y_n = \{y_0,\cdots,y_n| -1 = y_0 < y_1 < \ldots < y_n = 1\}$ be a partition of $[-1,1]$. Also, let $J_j = [y_j,\ y_{j+1}]$ with $y_j = -1, j<0$, and $y_j = 1, j>n$.

# 1 Auxiliary Results

In this section, we present some useful lemmas for our work.

**Lemma 1.1.** *(Remez inequality)[5] For any $q\in\Pi_r$ and a set $A$ s.t. meas$\{[-1,\ 1]\setminus A\}\leq s\leq\frac{1}{2}$, then $\|q\|_{C[-1,1]}\leq e^{5n\sqrt{s}}\|q\|_{C[A]}$.*

**Lemma 1.2.** *Any spline $s$ from $\mathcal{S}_r$ that satisfies $\psi_{\phi}\left(s\right)\leq k$, where $\phi$ is bounded, implies $\|s\|_{\phi,\infty}^{\omega_{\mu,V}}$ is bounded.*

**Proof.**
From definition, it is enough to prove that $\|s\|_{\phi}$ is bounded. Since $\phi$ is convex bounded Orlicz function, then

$$\int_I \phi\left(\left|\lambda s\left(x\right)\right|\right)dx \leq \phi\left|\lambda\right| \int_I \phi\left(\left|s\left(x\right)\right|\right)dx \leq \phi\left(\left|\lambda\right|\right)k \leq M$$

For some $M = \max\left\{1+\phi\left(\left|\lambda\right|\right)k,\quad k\right\}$ implies $\|s\|_{\phi,\infty}^{\omega_{\mu,V}}\leq M$ □.

## 2 Main Results

Now, we are ready to prove that the best approximation of splines and/or polynomials exists, direct theorem is given as follow

**Theorem 2.1.** *Let $z_n$ be the Chebychev partition of $I$ into $n$ subinterval, such that $|I_j| \sim \frac{1}{n}$ . Also, let $f \in B^{\omega_{\mu,V}}_{\phi,\infty}$, if there is a bounded spline $s \in \mathcal{S}_r(z_n)$, s.t. $\|f - s\|_I \leq c\omega_\phi(f,t)$, then there is a polynomial $p_n \in \Pi_n$, that satisfies*

$$\|f - p_n\|_I \leq \omega_\phi(f,t).$$

**Proof.**
Define $p_n \in \Pi_n(I_j)$, for some $j = 1, 2, \ldots, n$ such that $\|s - p_n\|_{I_j} \leq \omega_\phi(f,t)_{I_j}$, then by Lemma 1.1. and Lemma 1.2,

$$\begin{aligned}\|f - p_n\|_I &\leq \|f - s\|_I + \|s - p_n\|_I \\ &\leq \omega_\phi\left(f, \frac{1}{n}\right) + \|s - q_n\|_I + \|q_n - p_n\|_I \\ &\leq \omega_\phi\left(f, \frac{1}{n}\right) + C\|s - q_n\|_{I_j} + C\|q_n - p_n\|_{I_j} \\ &\leq C\omega_\phi\left(f, \frac{1}{n}\right) \quad \square\end{aligned}$$

Now, we study the lower bound of the degree of best approximation

**Theorem 2.2.** *Let $f \in B^{\omega_{\mu,V}}_{\phi,\infty}$, then $\omega_\phi\left(f, \frac{1}{n}\right) \leq cE_{n,\phi}(f)$*

**Proof.**
Define each polynomial in $I_j$, $j = 0, 1, \cdots, n$ as $q_{n,j} \in \Pi_n(I_j)$, as follow $q_{n,n} - q_{n,0} = q_{n,n} - q_{n,n-1} + (q_{n,n-1} - q_{n,n-2}) + \cdots + (q_{n,1} - q_{n,0})$.

For some $m \leq n$, suppose that $\|q_{n,n} - q_{n,m}\|_\phi \leq E_{m,\phi}(f)$

By Theorem 2 and Lemma 1.2, we have

$$\begin{aligned}\omega_\phi\left(f, \frac{1}{n}\right) &\leq \omega_\phi\left(f - q_n, \frac{1}{n}\right) + \omega_\phi\left(q_n, \frac{1}{n}\right) \\ &\leq C\|f - q_n\|_\phi + \omega_\phi\left(q_n, \frac{1}{n}\right) \\ &\leq C\|f - q_n\|_\phi + \omega_\phi\left(q_{n,n} - q_{n,n-1}, \frac{1}{n}\right) \\ &\leq C\left(\omega_\phi\left(f, \frac{1}{n}\right) + \sum_{m=2}^{n}(m+1)E_{m,\phi}(f)\right) \\ &\leq CE_{n,\phi}(f) \quad \square\end{aligned}$$

# 3 Conclusions

This paper dealt with constructing and finding a best approximation from the space of polynomials and/or splines. The space of study was the Besov-Orlicz space. The modulus of continuity was used to estimate the degree of best approximation for both upper and lower bounds.

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