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Outer-Hop Independent Vertex Cover of a Graph

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Abstract

In this paper, we initiate the study on an outer-hop independent vertex cover of a graph and we characterize this type of set in some special graphs and in the join of two graphs. Moreover, we derive some bounds and formulas of the parameter using the characterization results.

Key words and phrases: Vertex cover, outer-hop independent vertex covering set, outer-hop independent vertex cover number.
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1 Introduction

A vertex cover of a graph is a subset of vertices that covers all the edges in the graph. In other words, every edge in the graph is incident to at least one vertex in the vertex cover. The concept of vertex covering sets has practical applications in various fields such as network design, optimization, and resource allocation. The size of the smallest vertex cover is called the vertex cover number of the graph. Finding the minimum vertex cover or the vertex cover number of a graph is an important problem in graph theory. It has applications in various optimization problems such as facility location, scheduling, and resource allocation. In practical scenarios, finding a minimum vertex cover helps in minimizing costs or maximizing efficiency. Some studies related to vertex cover of a graph can be found in [9, 10, 11].

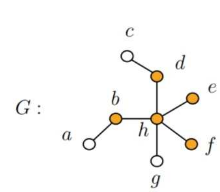
Recently, Bilar et al. [1] introduced and investigated the vertex cover hop domination in a graph and determined its relations with other parameters in graph theory. Moreover, they characterized the vertex cover hop dominating sets in some special graphs, join, and corona of two graphs and obtained the exact values or bounds of the parameters of these graphs. Some studies related to vertex cover hop domination and other hop-related concepts can be found in [2, 3, 4, 6, 7, 8].

In this paper, the outer-hop independent vertex cover of a graph is introduced and investigated on some graphs. The authors believed that this study would give additional information and would serve as reference for the future researchers who will study on variants of the vertex cover of a graph.

2 Main results

Definition 2.1. Let G be a graph. Then a non-empty set $P \subseteq V(G)$ is called an outer-hop independent vertex cover of G if $V(G) \setminus P$ is an empty or a hop independent set of G and for every edge of G is incidents to some vertex in P. The outer-hop independent vertex cover number of G, denoted by $\tilde{\beta}_{hi}(G)$, is the minimum cardinality among all outer-hop independent vertex covering sets of G.

Example 2.2. Consider the graph G below:



$P = \{b, d, e, f, h\}$. Then every edge of G is incident to some vertex in P. Thus P is a vertex cover of G. Observe that $d_G(a, c) = 4$, $d_G(a, g) = 3$ and $d_G(c, g) = 3$. It follows that $V(G) \setminus P = \{a, c, g\}$ is a hop independent set in G. Therefore, P is an outer-hop independent vertex cover of G. Moreover, it can be verified that $\tilde{\beta}_{hi}(G) = 5$.

Remark 2.3. *Let G be a graph and let $Q \subseteq V(G)$ be an outer-hop independent vertex cover of G. Then*

(i) Q is not necessarily a hop independent set of G;

(ii) $1 \leq \beta(G) \leq \tilde{\beta}_{hi}(G) \leq |Q| \leq |V(G)|$, where $\beta(G)$ is the vertex cover number of G; and

(iii) If Q is a minimum vertex cover of G, then $\tilde{\beta}_{hi}(G) = |Q|$.

Theorem 2.4. *Let G be a graph. Then $\tilde{\beta}_{hi}(G) \geq V(G) - \alpha_h(G)$, where $\alpha_h(G)$ is a hop independence number of G.*

Proof. Let G be a graph and let P be a minimum outer-hop independent vertex cover of G. Then $V(G) \setminus P$ is a hop independent set of G. Thus $\alpha_h(G) \geq |V(G) \setminus P| \geq |V(G)| - |P|$. It follows that, $|P| \geq |V(G)| - \alpha_h(G)$. Since P is the minimum outer-hop independent vertex cover of G, we have $\tilde{\beta}_{hi}(G) = |P| \geq |V(G)| - \alpha_h(G)$. □

Theorem 2.5. *Let $n \geq 2$ be a positive integer. Then $P \subseteq V(K_n)$ is an outer-hop independent vertex cover of K_n if and only if $|P| = n - 1$.*

Proof. Let $P \subseteq V(K_n)$ be a minimum outer-hop independent vertex cover of K_n. Then P is a vertex cover of K_n. Since $\beta(K_n) = n - 1$, it follows that $|P| \geq \beta(K_n) = n - 1$. Now, let $V(K_n) = \{a_1, a_2, \ldots, a_n\}$, and consider $Q = \{a_1, a_2, \ldots, a_{n-1}\}$. Then Q is an outer-hop independent vertex cover of K_n. Thus since P is a minimum outer-hop independent vertex cover of K_n, it follow that $|P| \leq |Q| = n - 1$. Therefore, $|P| = n - 1$.

Conversely, suppose that $|P| = n - 1$; say, $P = \{u_1, u_2, \ldots u_{n-1}\}$ where $V(K_n) = \{u_1, u_2, \ldots u_n\}$. Clearly, P is an outer-hop independent vertex cover of K_n. Since $\beta(K_n) = n - 1$ for all $n \geq 2$, it follows that P is a minimum vertex cover of K_n. Hence P is a minimum outer-hop independent vertex cover of K_n. □

Corollary 2.6. *Let $n \geq 1$ be a positive integer. Then*

$$\tilde{\beta}_{hi}(K_n) = \begin{cases} 1 & , \ \textit{if } n = 1 \\ n - 1 & , \ \textit{if } n \geq 2 \end{cases}$$

The following definition will be used to characterize the outer-hop independent vertex covering sets in the join of two graphs.

Definition 2.7. *Let G be a graph. Then $O \subseteq V(G)$ is called an outer-clique vertex cover of G if $V(G) \setminus O$ is an empty or clique in G and O is a vertex cover of G. The outer-clique vertex cover number of G, denoted by $\tilde{\beta}_c(G)$, is the minimum cardinality of an outer-clique vertex cover of G.*

Theorem 2.8. *Let G and H be two graphs. Then $N \subseteq V(G+H)$ is an outer-hop independent vertex cover of $G+H$ if and only if $N = N_G \cup N_H$ and satisfies one of the following conditions:*

(i) $N_G = V(G)$ and N_H is an outer-clique vertex cover of H.

(ii) $N_H = V(H)$ and N_G is an outer-clique vertex cover of G.

Proof. Suppose that N is an outer-hop independent vertex cover of $G+H$. If $N_G = V(G)$ and $N_H = V(H)$, then (i) and (ii) follow. Assume that $N_H \neq V(H)$. If $N_G \neq V(G)$, then there exist $x, y \in V(G+H)$ such that $x \notin N_H$ and $y \notin N_G$. However, $xy \in E(G+H)$, where $x, y \notin N$, a contradiction to the fact that N is a vertex cover of $G+H$. Thus $N_G = V(G)$. Since N is a vertex cover, N_H must be a vertex cover of H. Since $V(G+H) \setminus N = V(G+H) \setminus (V(G) \cup N_H)$ is a hop independent in $G+H$, $V(H) \setminus N_H$ must be a clique in H. Hence (i) holds. Similarly, (ii) holds.

The converse is clear. □

Corollary 2.9. *Let G and H be two graphs. Then*

$$\tilde{\beta}_{hi}(G+H) = min\{|V(G)| + \tilde{\beta}_c(H), |V(H)| + \tilde{\beta}_c(G)\}.$$

In particular, we have:

(i) $\tilde{\beta}_{hi}(K_n + P_n) = \begin{cases} 1 & , \text{ if } n = 1 \\ 2n-1 & , \text{ if } n \geq 2 \end{cases}$

(ii) $\tilde{\beta}_{hi}(K_n + C_n) = \tilde{\beta}_{hi}(P_n + C_n) = 2n - 1$ *for all* $n \geq 3$

(iii) $\tilde{\beta}_{hi}(F_n) = \tilde{\beta}_{hi}(K_1 + P_n) = n$ *for all* $n \geq 1$

(iv) $\tilde{\beta}_{hi}(W_n) = \tilde{\beta}_{hi}(K_1 + C_n) = n$ *for all* $n \geq 3$.

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